INFORMAL CALCULUS

Informal Calculus

With Applications to Biology and Environmental Science

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CONTENTS

WELCOME

Welcome to Informal Calculus! This book came from teaching Survey of Calculus at the University of Montana Western. You'll find that this book reflects the needs of that course, which are:

- Intuition behind the main concepts of calculus,
- Short, to-the-point explanations about how to do calculations,
- Build-in review for algebra along the way, with trigonometry not being a pre-requisite
- Applications to biology and environmental science, the two main science majors at my university.

A huge thanks to my collaborators for this project: Michelle Anderson who helped with biology ideas, problems, and projects; Rebekah Levine, who assisted with environmental science ideas, problems, and projects; and Debbie Seacrest, who advised on mathematical content and edited the entire book. This work was partially supported by a grant from TRAILS Montana. Thanks to Christina Trunnell for helping with many aspects of this book directly, as well as supporting OER projects all over Montana.

Click on "contents" to start exploring the book.

FOR TEACHERS

Feel free to take, change, modify any materials from this book for your own class or other use. If your particularly interested in project ideas, I've collected links to the projects in this book below:

- Ball toss
- Hard Derivatives
- Killdeer Migration
- Modelling with Differential Equations
- Measuring Streamflow
- Quake Lake

PHOTO CREDITS FOR THE COVER

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PART I

ALGEBRA TIPS AND TRICKS: PART I

CHAPTER 1

ALGEBRA TIPS AND TRICKS PART I (COMBINING TERMS, DISTRIBUTING, FUNCTIONS, GRAPHING)

Here are a few algebra tips and tricks to get you started. In later chapters, we will have some "just-in-time" algebra review, so you'll review an algebra concept just before you need it.

COMBINING LIKE TERMS

A *term* is one or more things multiplied together: for example, xyz is a term since it is x times y times z, b^2 is a term, since it is b times b. and x is a term. If there is also a number multiplied in front of a term, that is called the *coefficient* (if no coefficient is present, the coefficient is 1). Two terms are *like terms* if they have the same variables multiplied (but may have different coefficients). If two like terms are added together, they can be combined into one term by adding the coefficients.

Problem Combine like terms: $ab - a^2 + 2ab - 3a^2$.

$$\begin{aligned} ab - a^2 + 2ab - 3a^2 &= 1ab + (-1)a^2 + 2ab + (-3)a^2 && \text{(Clarify coefficients)} \\ &= 1ab + 2ab + (-1)a^2 + (-3)a^2 && \text{(Group like terms)} \\ &= (1+2)ab + (-1+-3)a^2 && \text{(Add coefficients)} \\ &= \boxed{3ab - 4a^2} \end{aligned}$$

DISTRIBUTING

If you are multiplying by a sum in parentheses, the rule is to *distribute*

$$a(b+c) = ab + ac$$

Here is another version in "table" form.

	b	$+c$
a	ab	$+ac$

It works, check it out:

$$\begin{aligned} 3(4+5) &= 3(4)+3(5) \\ 3(9) &= 12+15 \\ 27 &= 27 \end{aligned}$$

Here is an example:

Problem Distribute and combine like terms: $3a(2a-b)-(b-a^2)$.

$$\begin{aligned} 3a(2a-b)-(b-a^2) &= 3a(2a-b)+-1(b-a^2) \\ &= 6a^2-3ab-b+a^2 \qquad \text{(Notice the } +\mathbf{a^2}\text{)} \\ &= \boxed{7a^2-3ab-b} \end{aligned}$$

FOILING

When multiplying two sums, every term of the first must be multiplied by every term of the second. Thus, if there are two terms in the first sum and two in the second, there are four total terms in the product: the (f)irst two terms, the (o)utside terms, the (i)nside terms, and the (l)ast two terms. We can use the acronym "foil":

$$(a+b)(c+d) = ac+ad+bc+bd$$

Here is the same calculation in table form:

	c	$+d$
a	ac	$+ad$
$+b$	$+bc$	$+bd$

Here is an example:

Problem Foil: $(3a+4b)(a^2-ab)$.

$$\begin{aligned} (3a+4b)(a^2-ab) &= (3a)(a^2)+(3a)(-ab)+(4b)(a^2)+(4b)(-ab) \\ &= 3a^3-3a^2b+4a^2b-4ab^2 \\ &= \boxed{3a^3+a^2b-4ab^2} \end{aligned}$$

DISTRIBUTING WITH THREE TERMS

When you have three expressions multiplied together, things get a bit trickier. Let's do some examples.

Problem Find $(x-2)(x+1)(x+3)$.

To do this, we first multiply the $(x-2)(x+1)$. This is $x^2+x-2x-2 = x^2-x-2$. We then multiply $(x^2-x-2)(x+3)$. This is done by combining every term in the first product with every term in the last product. One way to do this is x times everything in x^2-x-2, plus 3 times everything in x^2-x-2.

$$\begin{aligned}(x-2)(x+1)(x+3) &= (x^2-x-2)(x+3)\\ &= (x^2-x-2)(x)+(x^2-x-2)(3)\\ &= x^3-x^2-2x+3x^2-3x-6\\ &= \boxed{x^3+2x^2-5x-6}\end{aligned}$$

There you go.

Alternatively, we can use the table method. We start by foiling two of the terms together

	x	-2
x	x^2	$-2x$
$+1$	$+1x$	-2

Adding the blue terms, we get an intermediate answer of x^2-x-2. Now we can multiply this by $x+3$.

	x^2	$-x$	-2
x	x^3	$-x^2$	$-2x$
$+3$	$+3x^2$	$-3x$	$+6$

Combining like terms gives the answer $\boxed{x^3+2x^2-5x+6}$, the same answer we got before!

Problem Find $(x+4)^3$.

We see this is the same thing as $(x+4)(x+4)(x+4)$. We then can do

$$\begin{aligned}(x+4)(x+4)(x+4) &= (x^2+8x+16)(x+4)\\ &= (x^2+8x+16)(x)+(x^2+8x+16)(4)\\ &= x^3+8x^2+16x+4x^2+32x+64\\ &= \boxed{x^3+12x^2+48x+64}\end{aligned}$$

I won't do it this time, but you could use the table method if you prefer that!

FUNCTIONS

A *function* is anything that produces an output for every possible input. So for example, $f(x) = 2x$ is the function that take in an input x, and outputs double x (i.e. $f(3) = 6$, $f(4) = 8$, $f(5) = 10$, etc.).

Here are some examples:

Problem If $g(x) = 2^x$, find $g(3)$ and $g(4)$.

We see that $g(3) = 2^3 = \boxed{8}$, and $g(4) = 2^4 = \boxed{16}$.

Problem If $h(x) = 2x + 3$, find $h(5)$, $h(y)$, and $h(x+1)$.

In each case, just replace the x with the input to the function. For example, $h(5) = 2(5) + 3 = 13$, and $h(y) = 2(y) + 3 = \boxed{2y + 3}$.

A tricky one is $h(x+1)$. Here, we replace the x with $(x+1)$ in the formula.

Tip: Always do substitutions or replacements like this in parentheses to keep it all together.

Here is what it would look like:

$$\begin{aligned} h(x) &= 2x + 3 \\ h(x+1) &= 2(x+1) + 3 \\ &= 2x + 2 + 3 \\ &= \boxed{2x + 5}. \end{aligned}$$

Problem If $m(x) = 3x - 1$, find $m(4x + 1)$.

We have to replace the x with $4x + 1$ in the formula. So we have

$$\begin{aligned} m(x) &= 3x - 1 \\ m(4x+1) &= 3(4x+1) - 1 \\ &= 12x + 3 - 1 \\ &= \boxed{12x + 2}. \end{aligned}$$

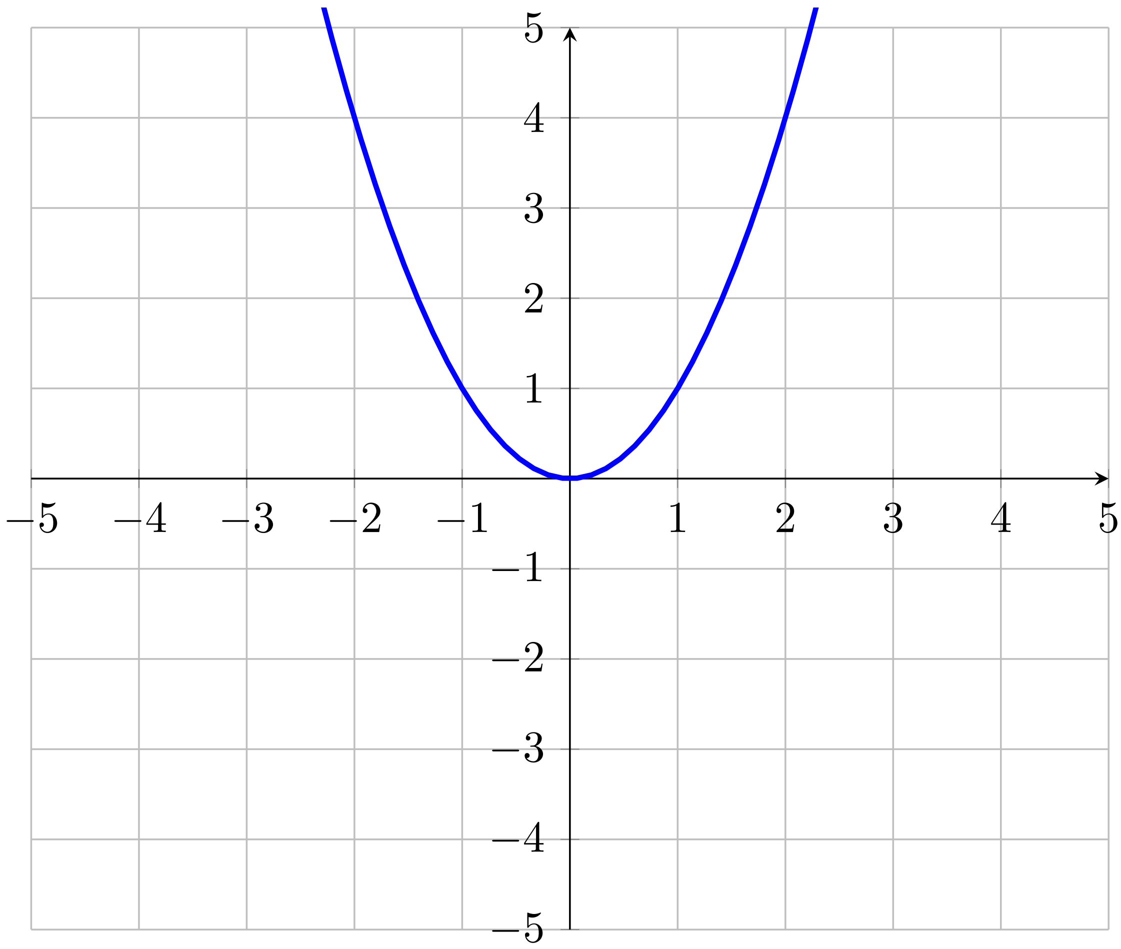
Problem If $f(x) = x^2 - 4x$ and $g(x) = 2x + 5$, what is $f(g(x))$?

Here, the idea is to replace x with $g(x)$ in the formula. In other words, x becomes $2x + 5$:

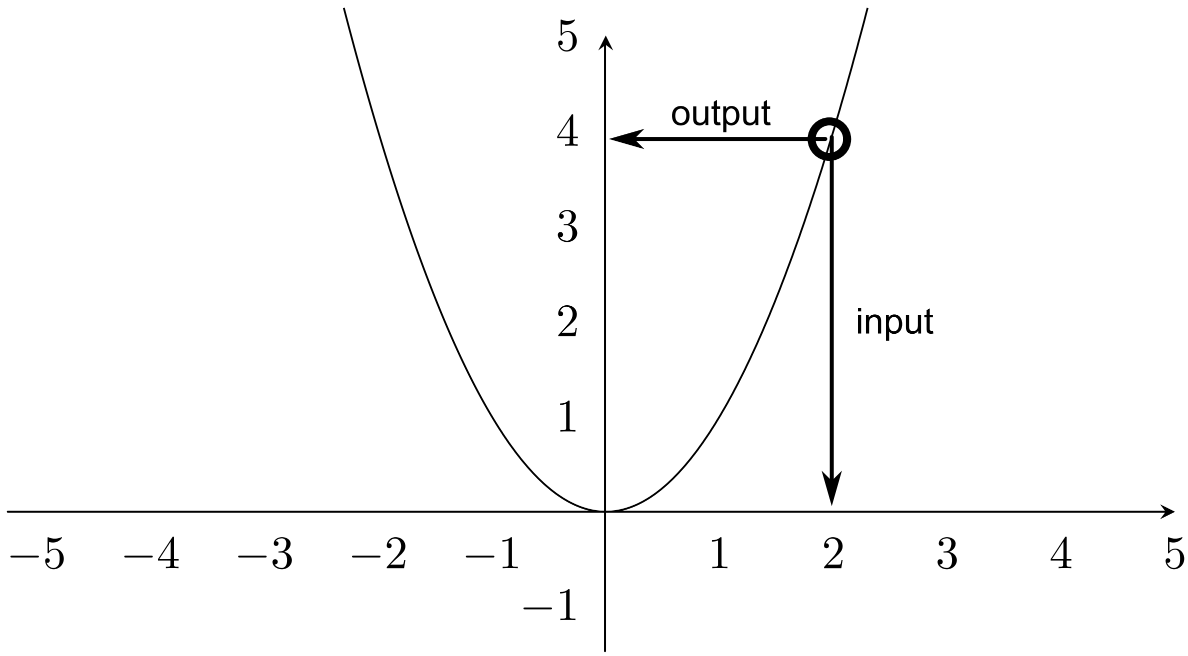
$$\begin{aligned}
f(x) &= x^2 - 4x \\
f(g(x)) = f(2x+5) &= (2x+5)^2 - 4(2x+5) \\
&= 4x^2 + 10x + 10x + 25 - 8x - 20 \\
&= \boxed{4x^2 + 12x + 5}.
\end{aligned}$$

GRAPHING FUNCTIONS

Graphing is a great way to visualize a function. For example, consider the graph of $f(x) = x^2$.



Choose any point on the curve. If you go down to the x-axis, you'll get the input value, and if you go directly left (or right) to the y-axis, you'll get the output value. For example,



This reflects the fact that $f(2) = 2^2 = 4$.

Note: Anything with **multiple outputs for one input** is considered **not** a function. A handy way to determine this is the "vertical line test" — any vertical line should hit a function only once.

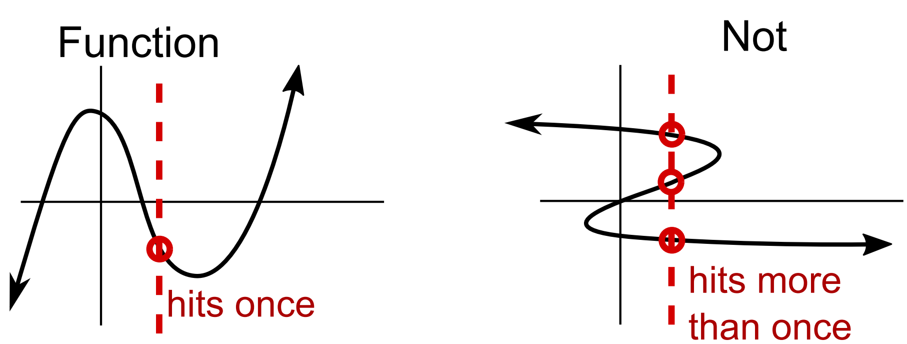


TABLE METHOD FOR GRAPHING

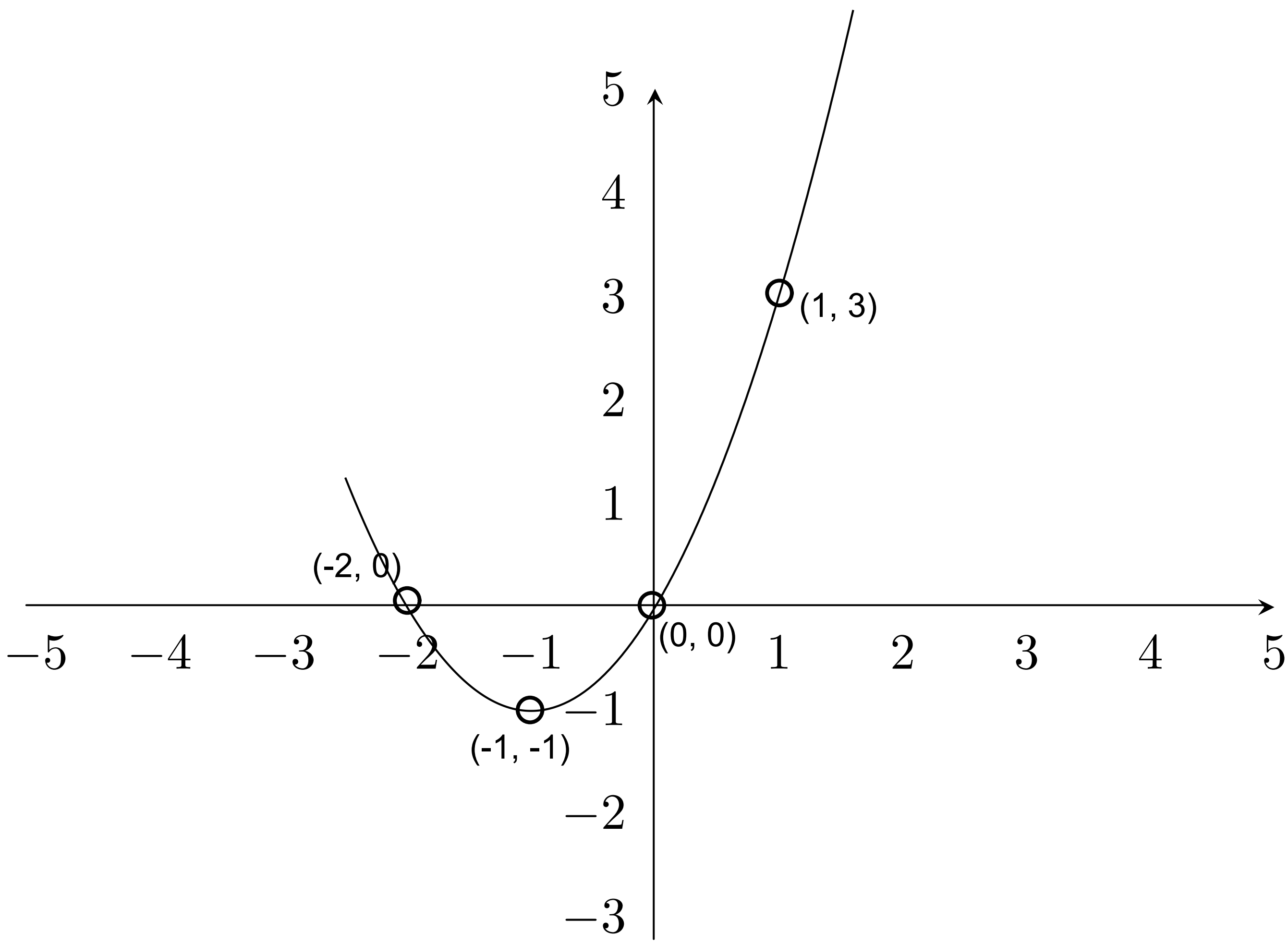
If you want to graph a function by hand, a way that works for virtually any function is the table method. Say we want to do the following:

Problem Graph $g(x) = x^2 + 2x$ using the table method.

We can just start by plugging in some values like $x = -2$, $x = -1$, $x = 0$, etc., and fill out a whole table. For example, for $x = -2$, we can compute $g(-2) = (-2)^2 + 2(-2) = 4 - 4 = 0$. Since $g(-2) = 0$, we know that the point $(x, y) = (-2, 0)$ lies on the graph. Filling out the rest of the table, we get

$\mathbf{x}$	$\mathbf{x^2+2x}$	$\mathbf{g(x)}$	$\mathbf{(x,y)}$
-2	$(-2)^2+2(-2)$	0	$(-2,0)$
-1	$(-1)^2+2(-1)$	-1	$(-1,-1)$
0	$(0)^2+2(0)$	0	$(0,0)$
1	$(1)^2+2(1)$	3	$(1,3)$
2	$(2)^2+2(2)$	8	$(2,8)$

We can then plot these input-output pairs on the graph, and they trace out a curve. (Note that the pair $(2,8)$ didn't fit on the graph.)



Chapter 2

HOMEWORK FOR ALGEBRA TIPS AND TRICKS: PART I

1. Simplify the following algebraic expressions.

   a. $(y+2)^2 - (y-1)^2$
   $6y+3$
   ans
   b. $\frac{5x^3+6x^2+x}{x}$
   $5x^2+6x+1$
   ans
   c. $\frac{(x-3)^2-9}{x}$
   $x-6$
   ans

2. Simplify the following.

   a. $(x-3)(x+1)(x-2)$
   x^3-4x^2+x+6
   ans
   b. $(x+1)^3$
   x^3+3x^2+3x+1
   ans
   c. $(x+h)^3$
   $x^3+3x^2h+3xh^2+h^3$
   ans
   d. $(x-1)^3-x^3-1$
   $-3x^2+3x-2$
   ans

3. Given the functions $f(x)=5x-10$ and $g(x)=3x+4$, find the following.

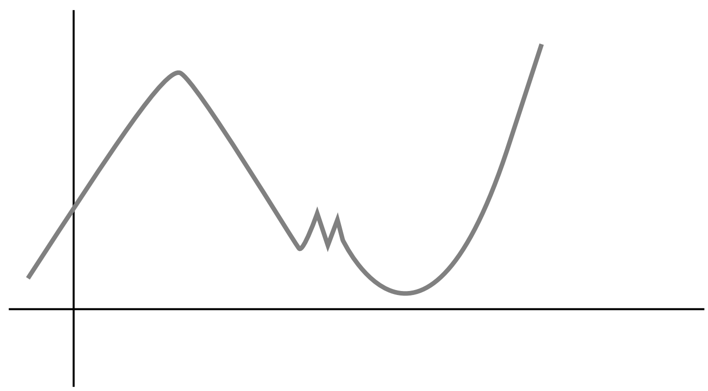
   a. $f(7)$
   25
   ans
   b. $g(4)$
   16
   ans
   c. $f(x+3)$

$5x + 5$
ans

d. $g(3y - 2)$
$9y - 2$
ans

e. $f(g(x))$
$15x + 10$
ans

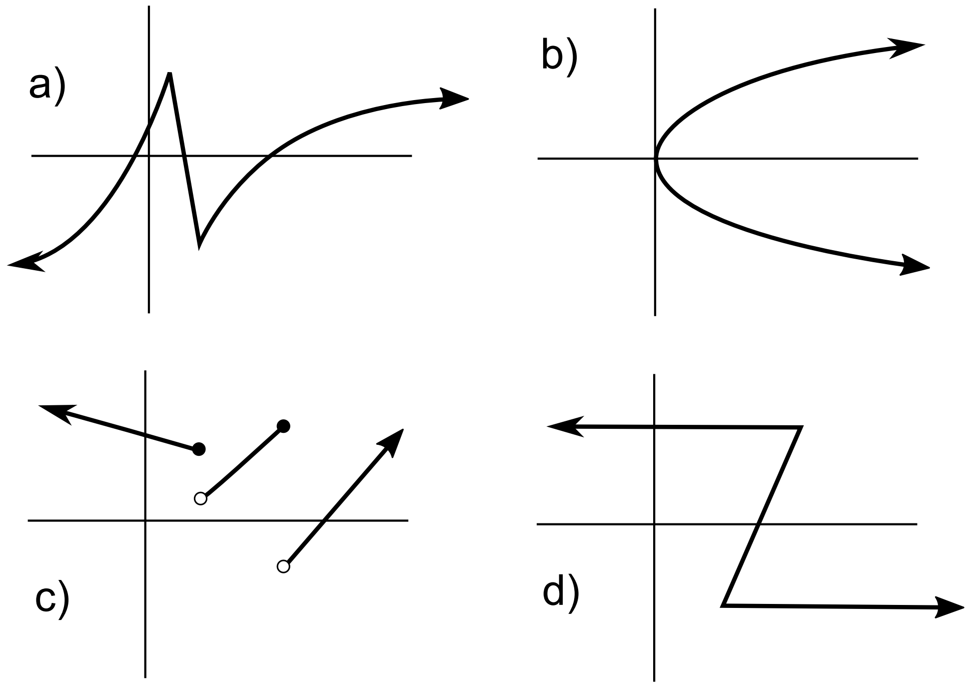
4. Sketch graphs of e^x, $\frac{1}{x}$, and $\sin(x)$. (if you don't know what it looks like, use a calculator or look up the answer on the internet).
5. For the following graph, imagine that it represents the amount of money in Rebecca's bank account. Create a story that explains the various ups and downs.



Answers vary. Here is one possibility (please don't use this, come up with your own!): Rebecca is a money counterfeiter. Business is booming in the 1990s, and she makes (literally) a boat load of cash, launders it, and makes bank. However, the Feds in the 2000s came out with these new benjamins with watermarks and stuff like that, and she can't counterfeit it anymore. She loses all her money gambling on water polo. She tries to get her business back a couple times, but it never catches on. Finally, she decides to invest in bitcoin when it was trading at $1 USD per bitcoin, and then she made a serious fortune.
ans
6. 0° in Celsius is 32° in Fahrenheit, and 100° in Celsius is 212° in Fahrenheit.

   a. Sketch a graph with Fahrenheit along the x-axis and Celsius along the y-axis. Hint: I'd start with the points $(32, 0)$ and $(212, 100)$, and connect them with a straight line.
   b. What is the slope of the graph from part (a)?
   c. What is a formula to convert from Fahrenheit to Celsius?

7. For each graph below, state whether it is a function or not. (This involves using the "Vertical Line Test")



A and C are functions, B and D are not.
ans

8. Let $f(x) = 15 - 2x^2$.

   a. Find the slope of a line that goes through the points $(-1, 13)$ and $(0, 15)$, both of which lie on the graph of $f(x)$.
   Slope is 2
   ans

   b. Consider the function $f(x) = 15 - 2x^2$. Find the slope of a line that intersects this curve at $x = 1$ and $x = 2$.
   Slope is -6
   ans

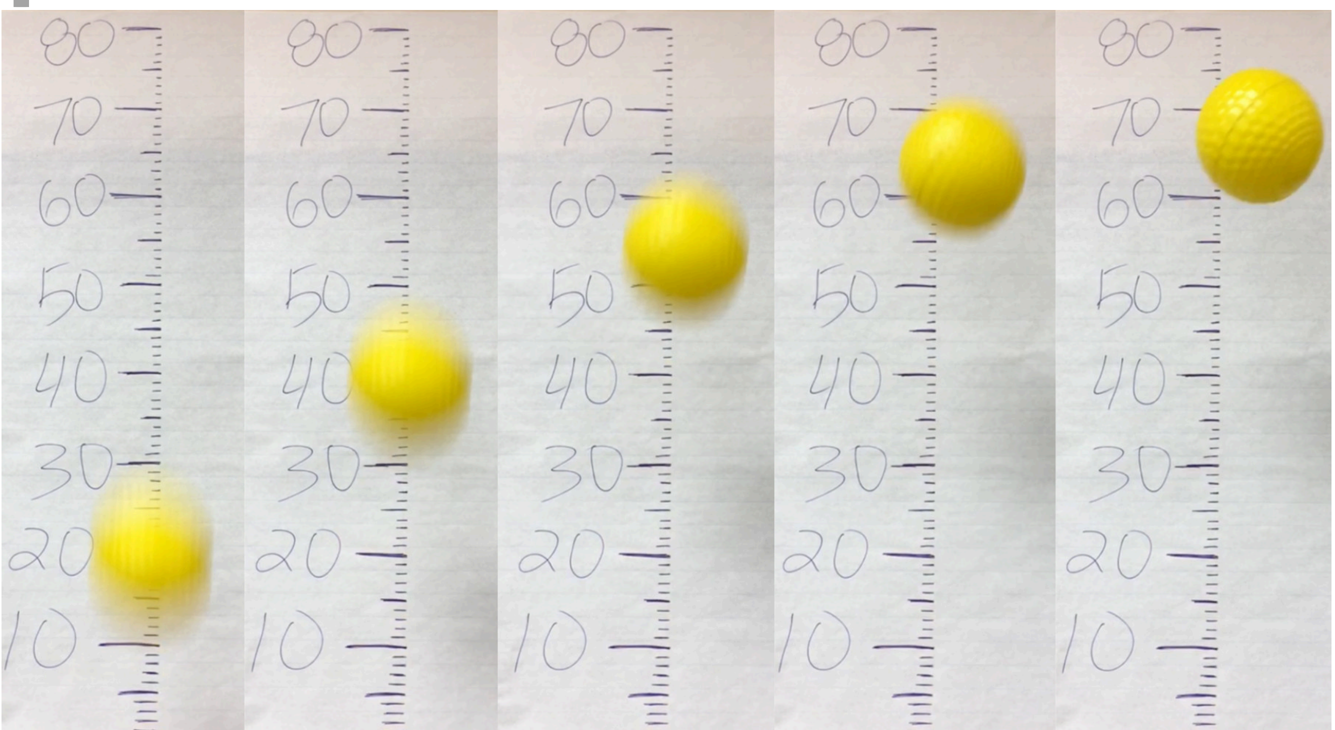
PART II

DERIVATIVE INTRODUCTION

Chapter 3

Ball Toss Project

Purpose of the project: get familiar with the idea of creating a velocity function based on position data



Above are frames from a video of a simple ball toss. Every other frame from the 30 frames per second video is shown, so the time between the frames is approximately $1/15$ of a second.

1. Fill in the position data for each time in the table below. Velocity is usually defined as $\frac{\text{Position}}{\text{Time}}$. How could you calculate velocities for each data point? Fill these in as well.

Time (s)	Position (cm)	Velocity (cm/s)
0		
$\frac{1}{15}$		
$\frac{2}{15}$		
$\frac{3}{15}$		
$\frac{4}{15}$		

2. Create two graphs, either by hand or with graphing software: position versus time, and velocity versus time. In each case, time is the x-axis, while the y-axis is position in the first graph and velocity in the second graph. How are the two graphs related?

   Going from a position graph to a velocity graph like this is called a *derivative*, which we'll talk a lot more about in upcoming chapters.
3. While your velocity graph is probably not a perfect line, imagine that it is linear for a second. What is the slope of the velocity graph? This is the rate of acceleration due to gravity, an important quantity in physics.

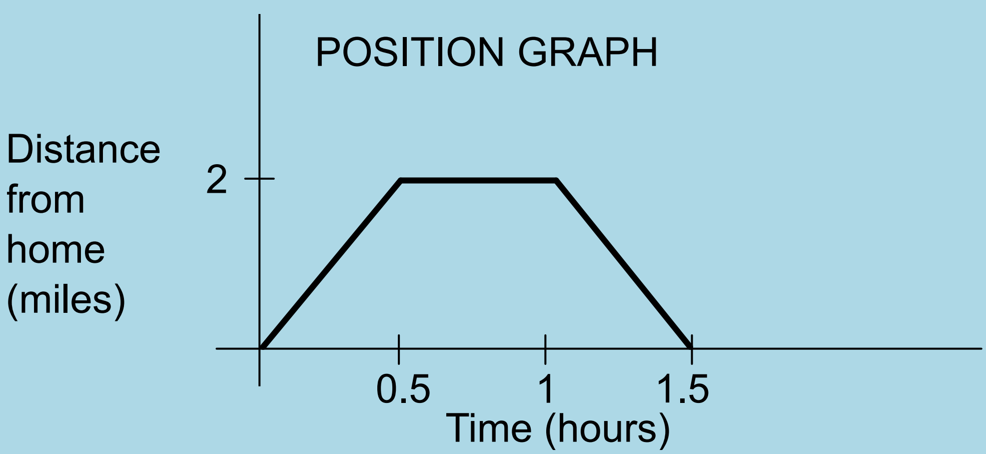
Chapter 4

POSITION TO VELOCITY

The idea of position of an object versus the velocity of an object encompasses all the big ideas of calculus. So that's where we'll start!

HEADING TO A LAKE

Problem Suppose you're given a graph of your distance from home during a trip to the lake. It might look something like this:



This graph might represent you walking to a lake two miles away, hanging out for half an hour, then walking home. The first part of the graph that slants upwards represents your walk to the lake, since your distance from home is increasing (higher on the graph). The second part of the graph represents you hanging out at the lake. It's flat since your distance from home is not changing. Finally, the part of the graph that slants down represents you walking home. Your distance to home is decreasing, so the line goes down on the graph.

Now here is the question: what is your *velocity* during this journey?

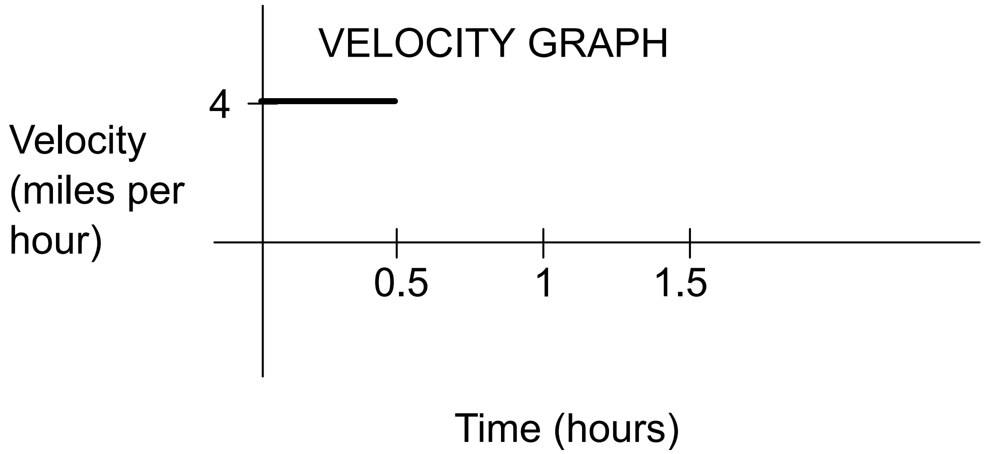
Velocity is a measure of speed, and essentially boils down to this equation:

$$\text{Velocity} = \frac{\text{change in distance}}{\text{change in time}}.$$

While you're walking to the lake, you're traveling at a rate of 2 miles every half hour (your change in distance is two, during the half hour change in time). Therefore your velocity is $\frac{2}{\frac{1}{2}}$. We can simplify this fraction by multiplying top and bottom by 2, and we see

$$\text{Velocity to the lake} = \frac{2}{\frac{1}{2}} \cdot \frac{2}{2} = \frac{4}{1} = 4.$$

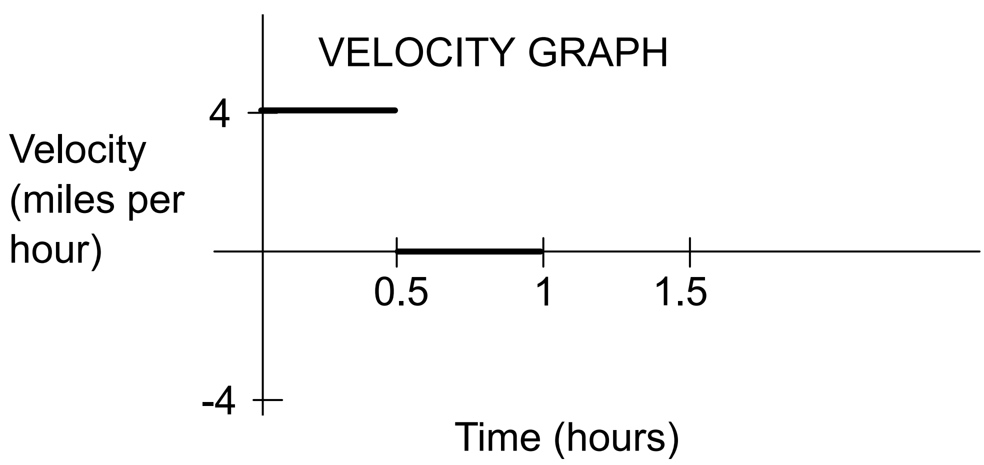
So you were walking at 4 miles per hour to the lake. This velocity doesn't change as you walk to the lake, and so we call this a *constant velocity*. Graphically, we represent constant velocity with a horizontal line:



While at the lake, your position is not really changing 'cause you're just hanging out. So your velocity is zero. To relate this back to the formula, your change of distance is zero while your change in time is $1/2$. So by the formula we have

$$\text{Velocity at the lake} = \frac{0}{\frac{1}{2}} = 0,$$

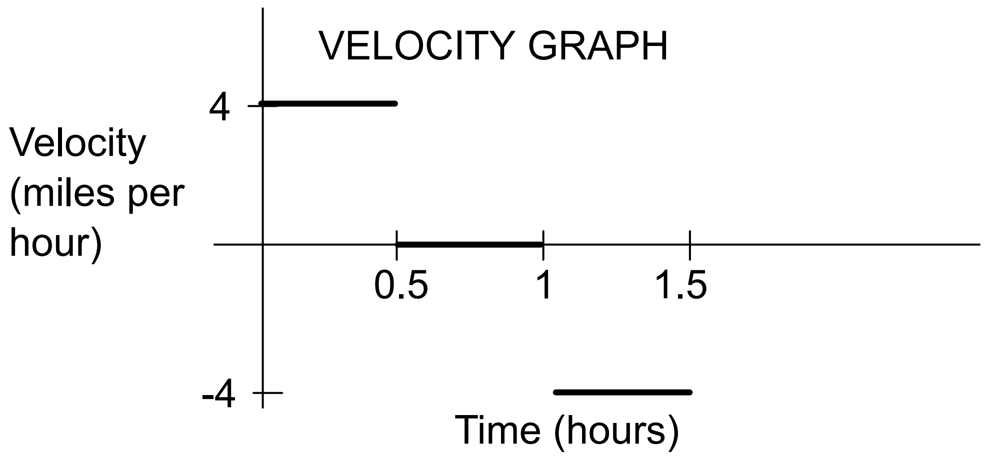
but we didn't really need to do the formula since we knew we weren't going anywhere. Graphically, a velocity of zero looks like this:



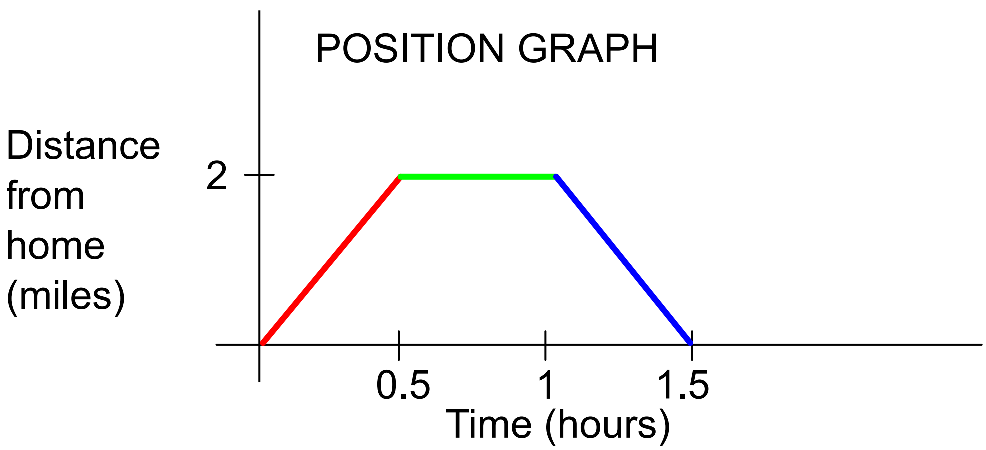
Finally, while coming back home, again we have a change in distance of 2 miles over a half an hour. So you might think the velocity is 4 again, but it is actually very natural to call this a *negative velocity*, since the distance is going down. So we say that the change in distance is actually -2, and therefore:

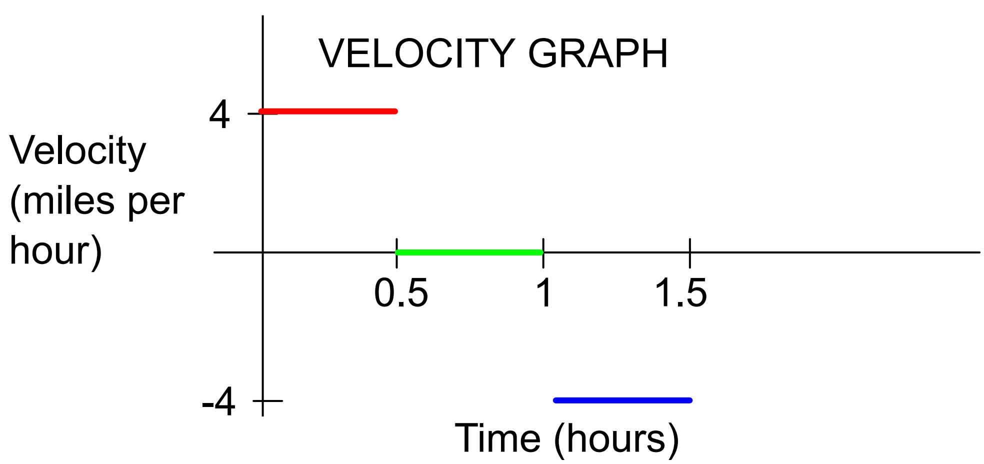
$$\text{Velocity returning home} = \frac{-2}{\frac{1}{2}} \cdot \frac{2}{2} = \frac{-4}{1} = -4,$$

Graphically:



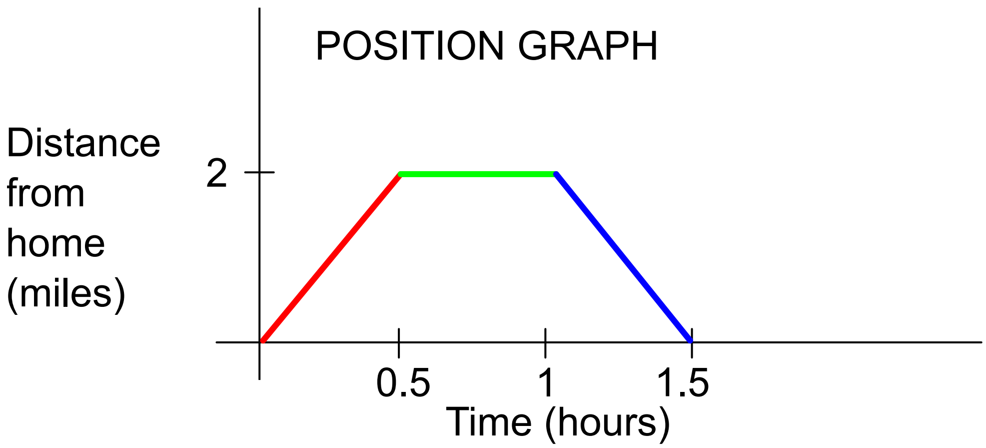
Okay, now let's look at the position and velocity graphs together. I'll color each segment to emphasize how the different parts correspond.

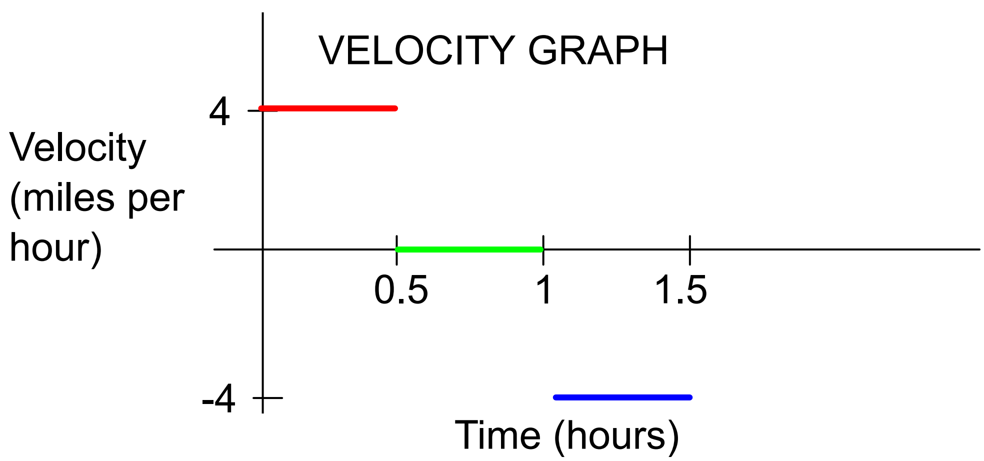




SLOPE

Another way to think of velocity is that it is the same as the slope of a line. Recall that the slope of a line is a measure of how steep the line is, and the formula follows the phrase "rise over run". Let's look again at the position and velocity graphs from the last subsection:





What is the slope of the red line? Well, rise over run would be $\frac{2}{0.5}$, which is 4. That's the same as the velocity graph! Same thing for the green line: it has a slope of zero, and the velocity graph is at zero. Finally, the slope of the blue line is $\frac{-2}{0.5}$ which is -4, and that is what we have for the velocity graph.

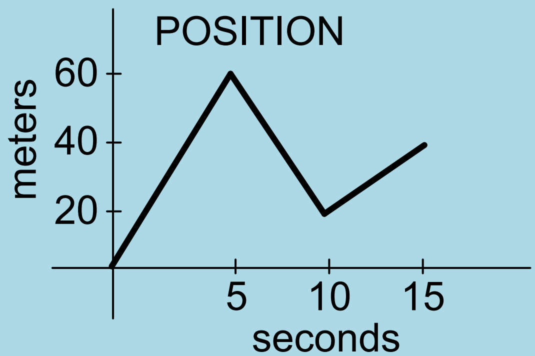
So "slope" and "velocity" are the same thing. But there is another name for this concept that we will use a lot: **derivative**. Derivative, slope, and velocity all mean the same thing.

OTHER EXAMPLES OF DERIVATIVES

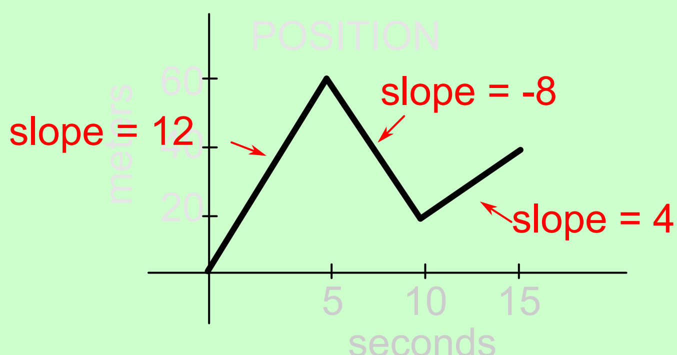
Let's see some other examples. Note for each of these, the position graphs is always *piecewise linear*, or made up of line segments. This makes it easier to find the velocity, or slope.

Example **Position to Velocity**

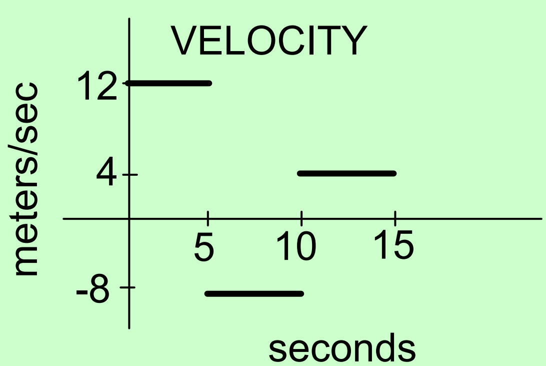
Problem Find the velocity graph (i.e. the derivative) corresponding to the following position graph.



To solve this problem, we need to find the velocity, or slope, of each of the lines in the graph. The first line has a change of distance of 60, and a change of time of 5 seconds, so the velocity is $60/5 = 12$. Next, the we have a negative change of distance of -40 since the graph goes from 60 to 20. This also occurs over 5 seconds, so the velocity is $-40/5 = -8$. Finally, we gain a distance of 20 in the final line over five seconds, so the velocity is $20/5 = 4$.



If we graph these velocities, we have

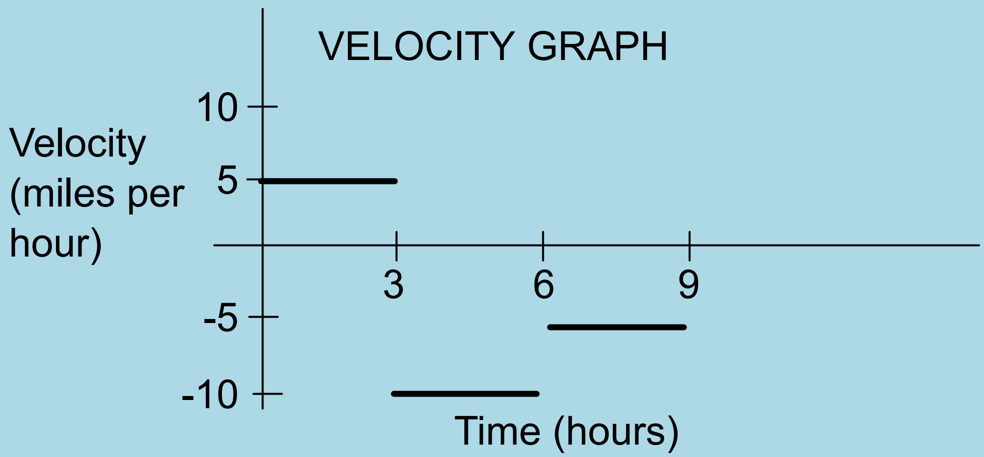


INTEGRALS

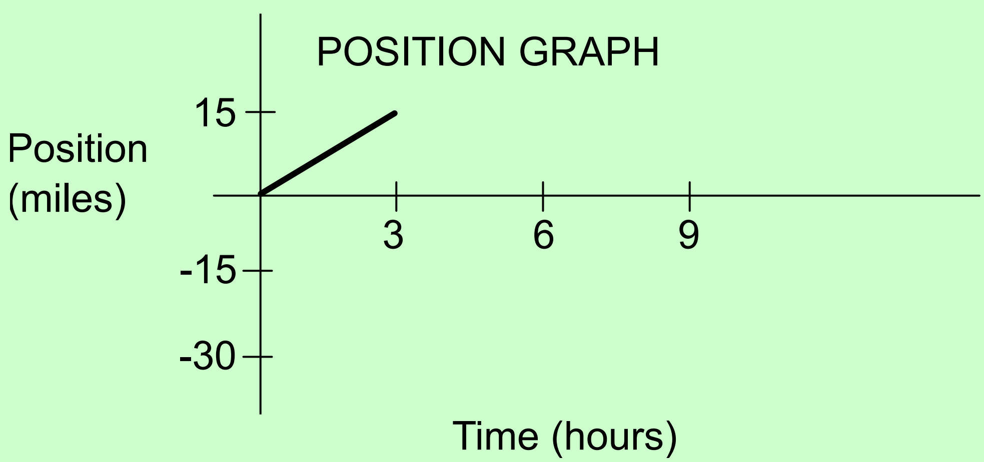
We can also go in the reverse direction: take a velocity graph, and create a position graph. This is called *integration* or *taking an integral.* This can be tricky but we can do it at this point if the function is what is called a *step function*, which is basically a function consisting of a bunch of flat parts.

Example **Integration**

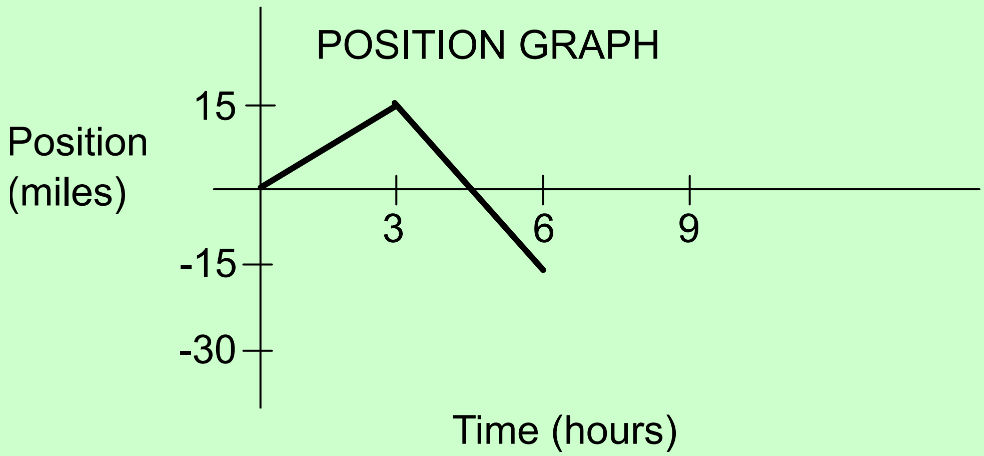
Problem Given the following velocity graph, create a position graph.



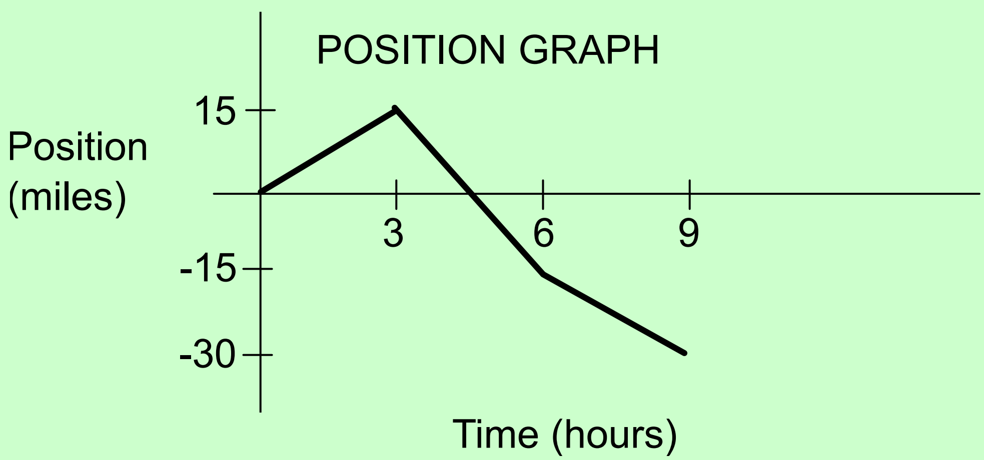
Given these velocities, we want to graph where the person travels, and the person can start wherever we want. For convenience we will start the person at location zero. If we focus on the first section, we see the person is traveling at 5 miles per hour for three hours. This corresponds to the person traveling 15 miles total during the first three hours. It would look something like this:



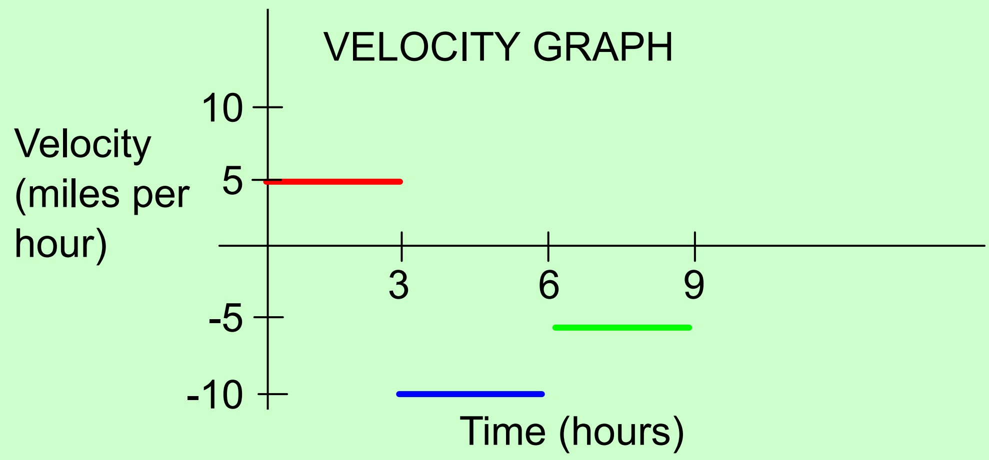
Notice how the position goes from 0 to 15. From 3 to 6, the person moves at -10 miles per hour for 3 hours: this would be a total of -30 miles traveled. Since the person is already at position 15 , they'll end up at position $15 + (-30) = -15$.

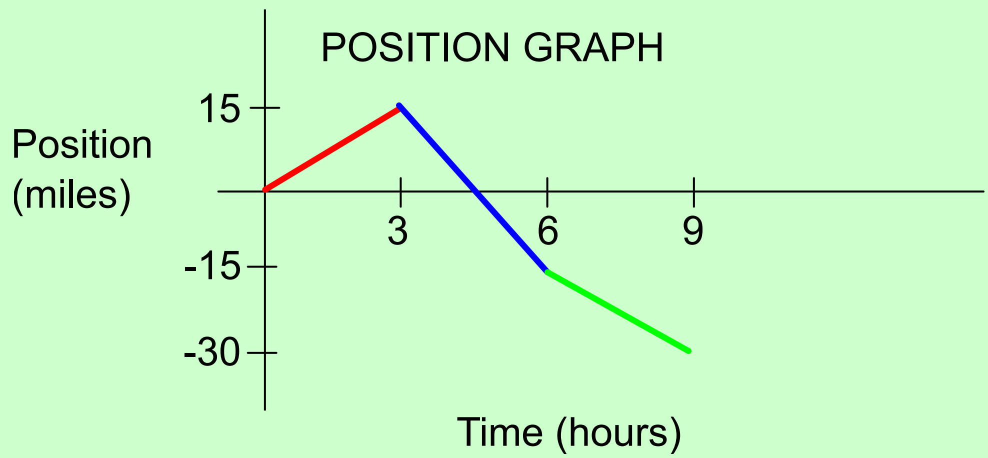


From 6 to 9, the person moves at -5 miles per hour for 3 hours, which is another -15 miles covered. Starting from position -15 and adding another -15, the person will end up at -30.



Finally, here are colored versions of the velocity and position maps.

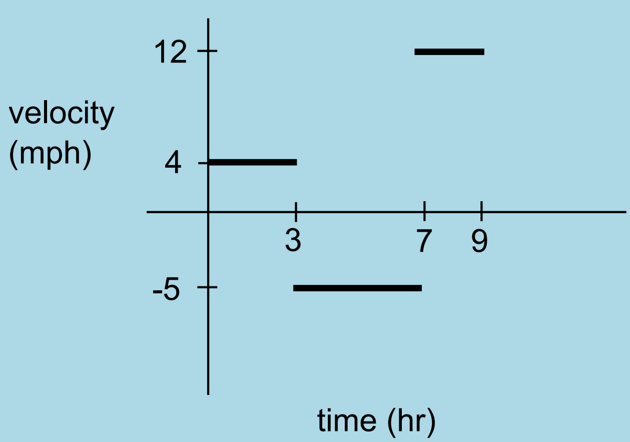




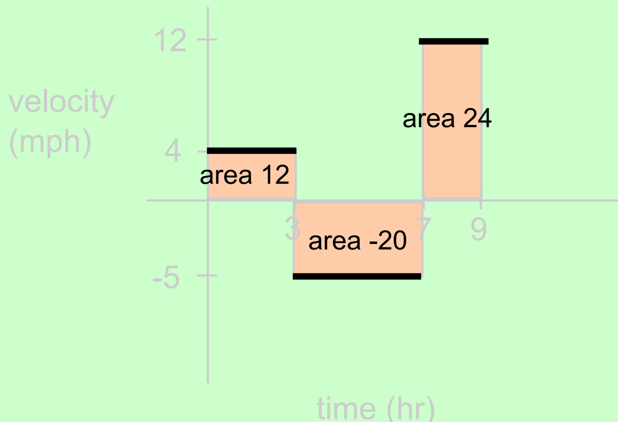
Going from velocity back to position is called an **integral**. Here is another example.

Example **Integration**

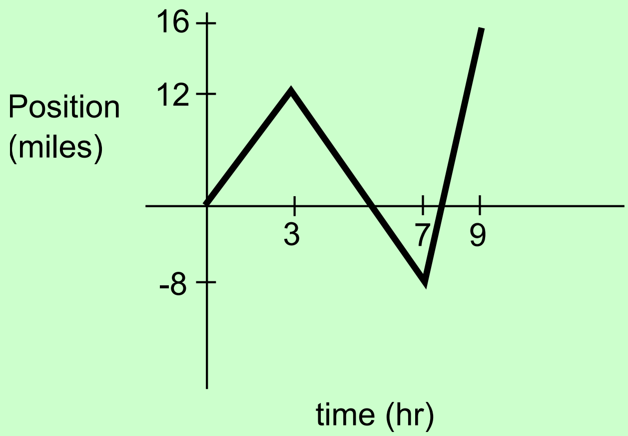
Problem Given the following velocity graph, create the position graph.



To find out the position, note that we use multiplication. For example, in the first 3 hours, they move at 4 miles per hour, so we multiply: $3 \cdot 4 = 12$ miles. In the next 4 hour stretch, we're at -5 mph, so we multiply $4 \cdot (-5) = -20$. The last bit is two hours long at 12 mph, so we multiply $2 \cdot 12 = 24$. Note that we ALSO multiply when we find area, so we can think of these calculations (velocity to position) as area calculations:



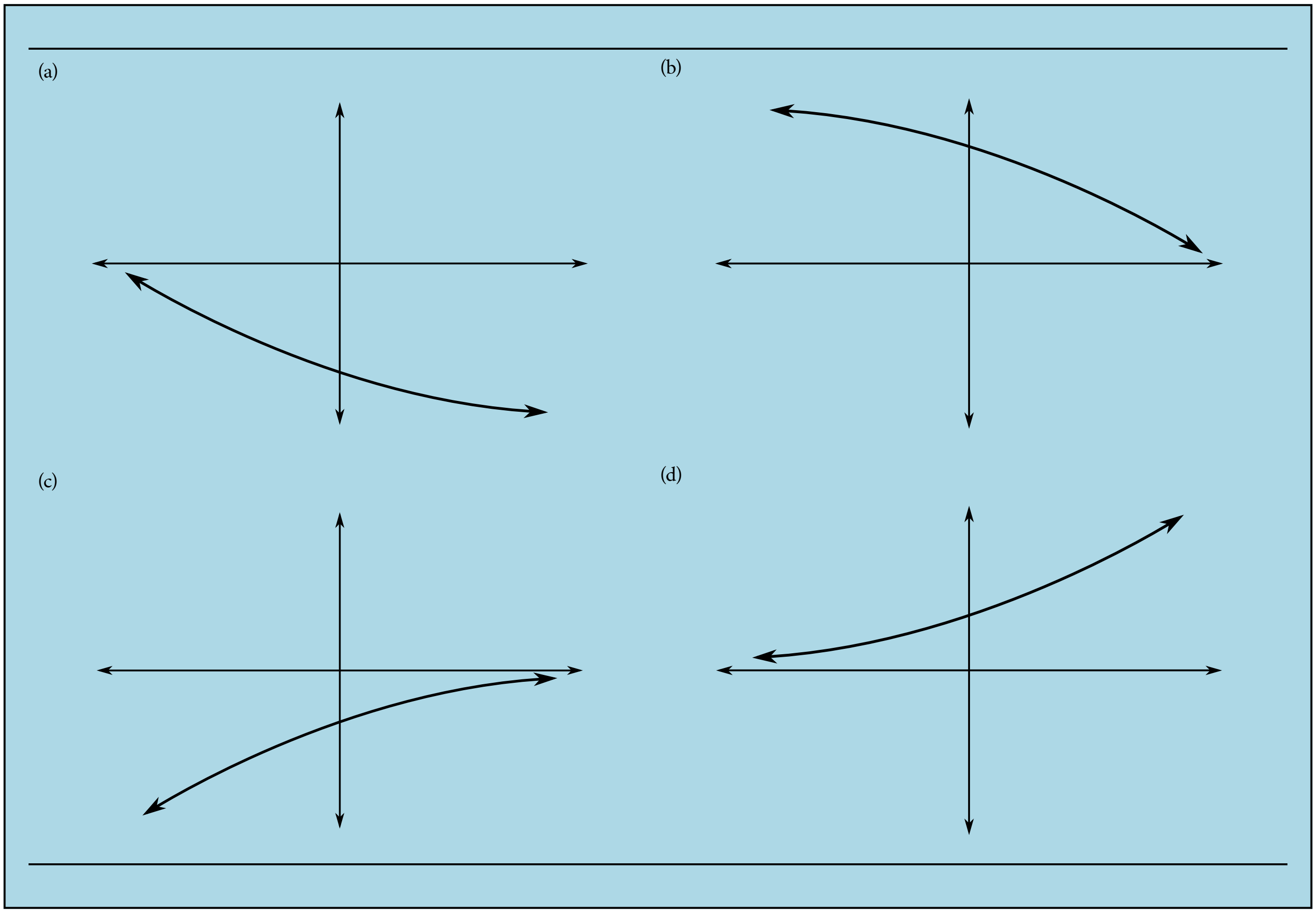
So we basically make jumps of 12, -20, and 24 as shown:



INCREASING AND DECREASING

The following graphs are not made up of straight lines — but we can still tell if the derivative is positive or negative. A *positive derivative* means a quantity is increasing — and graphically that is represented by a graph the climbs as you go from left to right. A *negative derivative* means a quantity is getting smaller — graphically going downward from left to right.

Problem Which of the following have positive derivatives? Which have negative derivatives?



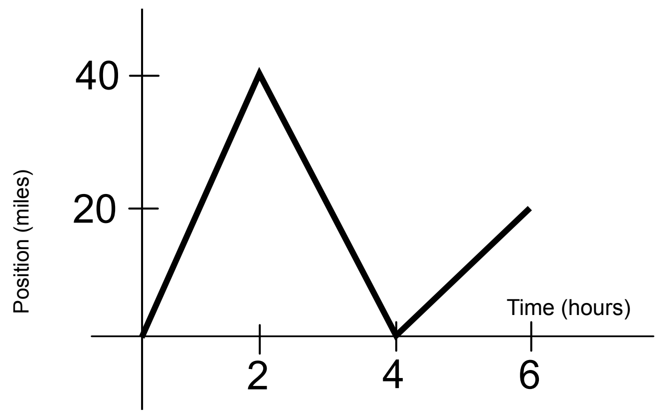
That's right: (a) and (b) have a negative derivatives, and (c) and (d) have positive derivatives.

CHAPTER 5

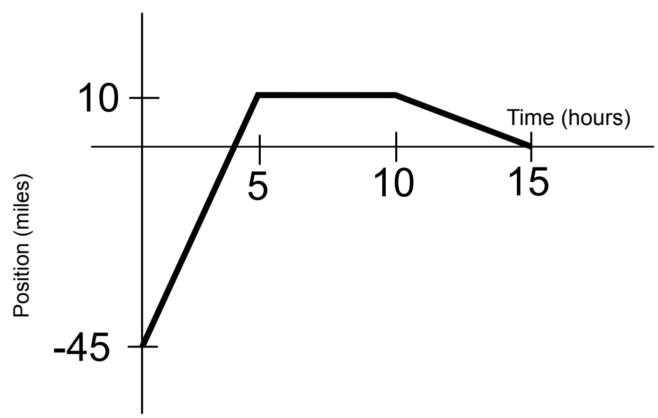
HOMEWORK: POSITION TO VELOCITY

1. For each position graph, sketch the velocity graph. (This process is known as "taking the derivative".)

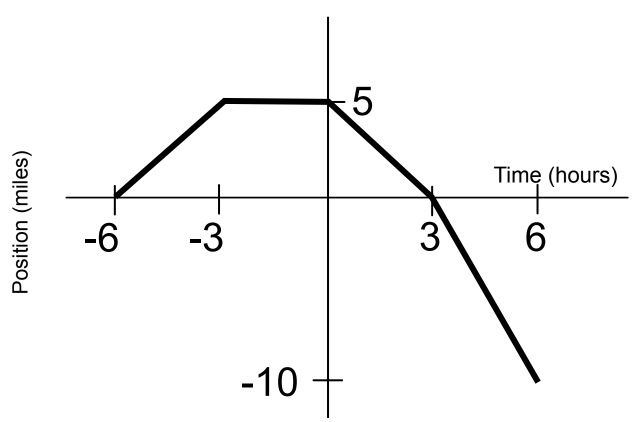
a.



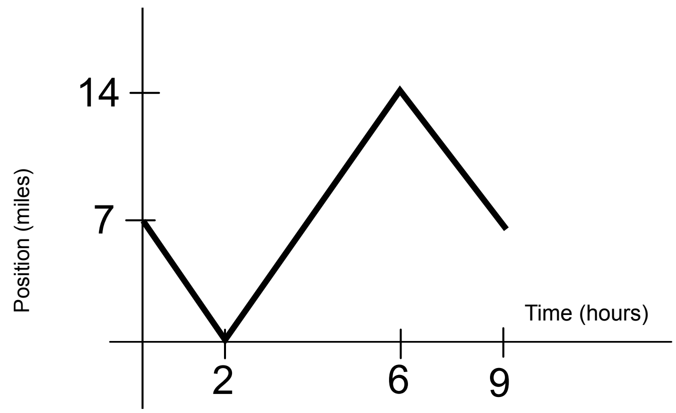
b.



c.

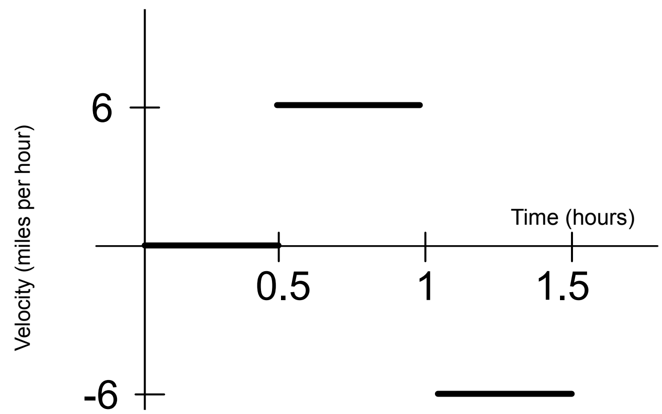


d.

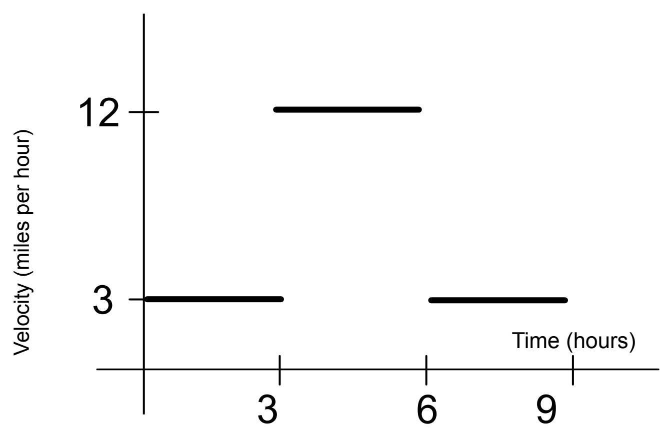


2. For each velocity graph, sketch the position graph. (This process is known as "integrating".)

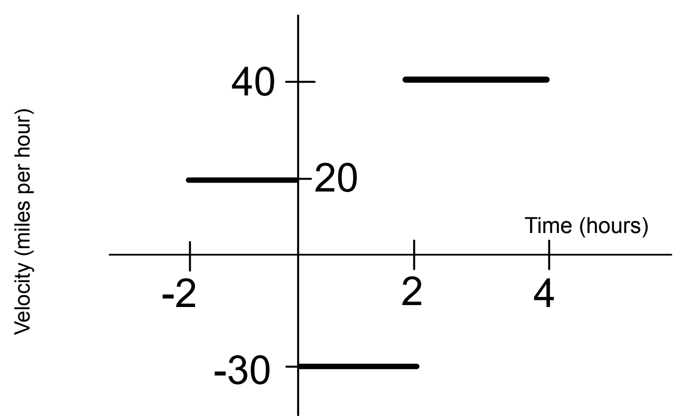
a.

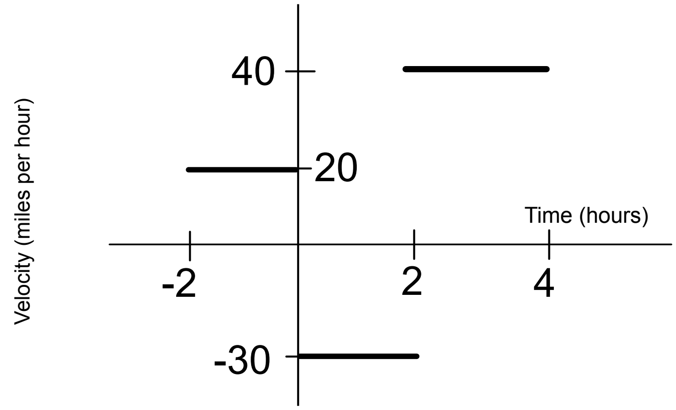


b.



c.



d. 

3. The Consumer Price Index (CPI) is a number that correlates to how expensive it is to buy things. Suppose the CPI is increasing. Circle one of the bold choices for each problem.

   a. The CPI has **positive** or **negative** derivative.
   b. The CPI has **positive** or **negative** slope.
   c. Using the CPI as an indicator, consumer prices were **higher** or **lower** yesterday.
   d. Using the CPI as an indicator, consumer prices will be **higher** or **lower** tomorrow.

   Positive, positive, lower, higher
   ans
4. The Beaverhead River Flow Rate (BRFR) measures how much water is flowing in the Beaverhead river. Suppose the BRFR has negative slope. Circle one of the bold choices for each problem.

   a. The BRFR has **positive** or **negative** derivative.
   b. The BRFR is **increasing** or **decreasing**.
   c. There will be **more water** or **less water** flowing in the river tomorrow.
   d. There was **more water** or **less water** flowing in the river yesterday.

   negative, decreasing, less water, more water
   ans
5. The temperature of a chemical sample has a negative derivative. Circle one of the following bold options.

   Initially the sample was at room temperature. Then the sample was put in an **oven** or **refrigerator**.
   Refrigerator
   ans
6. Acceleration is the measure of how quickly the velocity is changing. Suppose the velocity of a car is positive, but the acceleration is negative. Circle the bold option in each case.

   a. The position of the car is **increasing** or **decreasing**.
   b. The velocity of the car is **increasing** or **decreasing**.
   c. If the current trends continue, in five seconds the car will have a **greater** or **smaller** position number.
   d. If the current trends have held true for the last five seconds, five seconds ago the car was going **faster** or **slower**.

Increasing, decreasing, greater, faster
ans

7. For each situation, try to sketch a picture of a graph matching the description.

    a. A graph with negative values (when I say values, I mean y-values!).
    b. A graph with a positive derivative (when I say derivative, think slope!)
    c. A graph with negative derivative.
    d. A graph with positive derivative but negative values.
    e. A graph with negative acceleration.
    f. A graph with negative acceleration but positive derivative.

CHAPTER 6

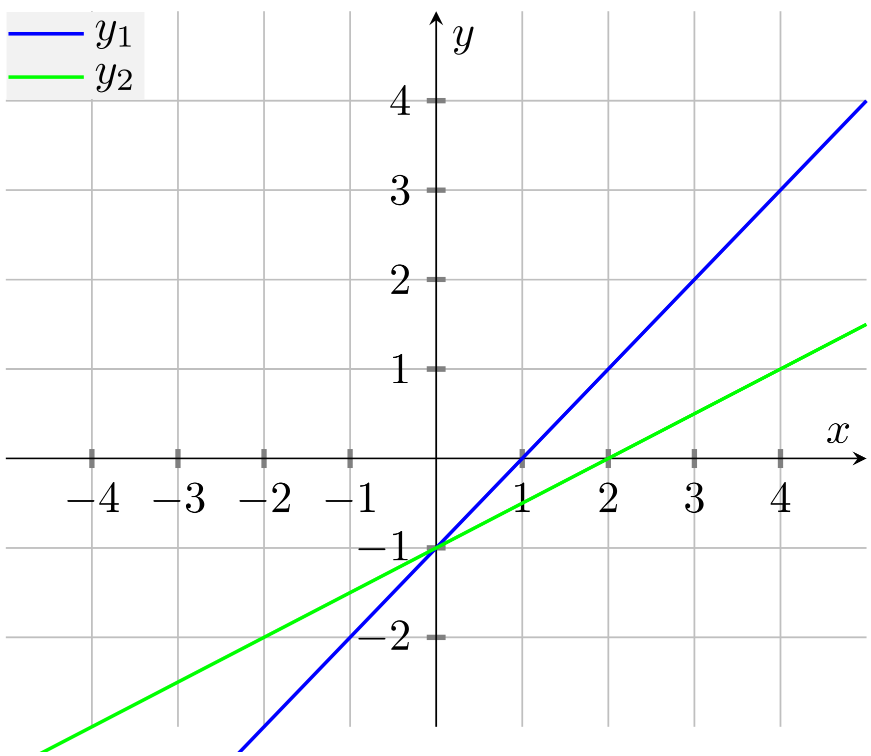
ALGEBRA TIPS AND TRICKS PART II (PIECEWISE DEFINED FUNCTIONS)

PIECEWISE DEFINED FUNCTIONS

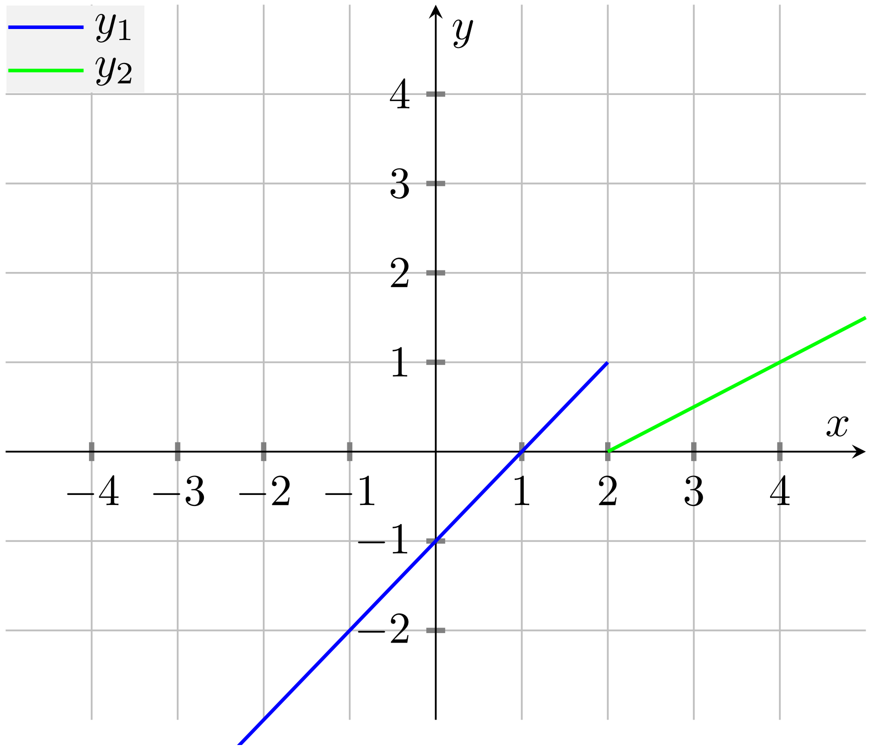
Problem Graph the following function

$$g(x) = \begin{cases} x-1 & x \leq 2 \\ \frac{1}{2}x - 1 & x > 2 \end{cases}$$

How do you do it? Well, you have to graph two different lines: $y_1 = x - 1$ and $y_2 = \frac{1}{2}x - 1$:



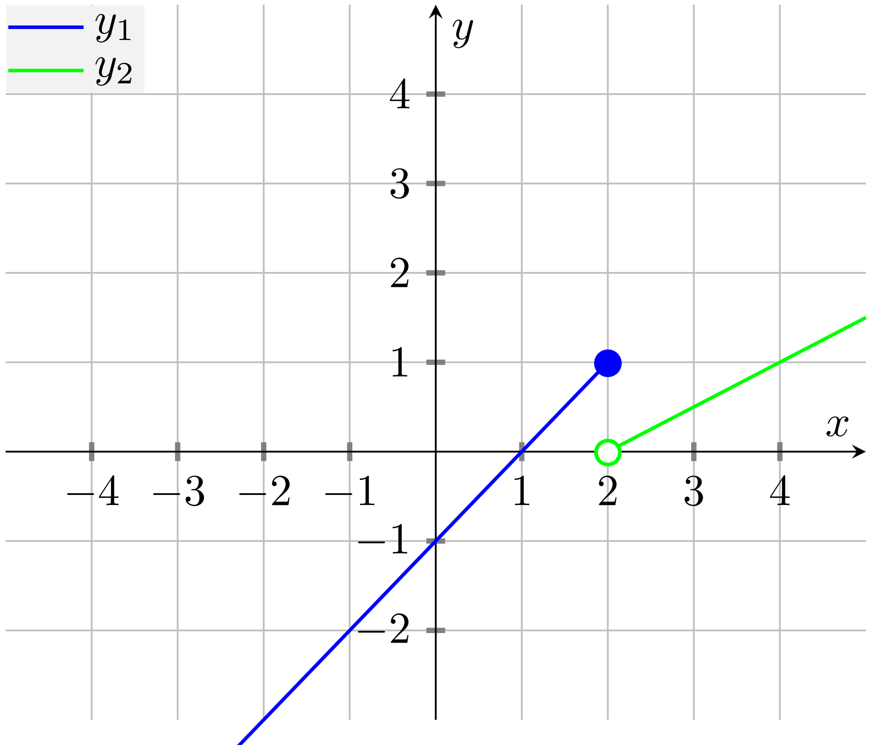
But then you need to "cut off" the graph of y_1 after $x = 2$, and "cut off" the graph of y_2 before $x = 2$
:



That's the graph of $g(x)$! It is called a *piecewise defined function*. Since each piece is linear, sometimes it is called a *piecewise linear function*.

There is one more detail to clear up. What is the value of the function at $x = 2$?

Well, going back to the original function, we see that $g(x)$ was defined as $x - 1$ for $x \leq 2$, and this includes $x = 2$. So we should use the blue line to determine the y-coordinate for $x = 2$. To indicate this on the graph, a filled in dot can be added to the blue graph (indicating the endpoint is included), and an open or not-filled-in dot is added to the green graph (to indicate the endpoint is not included).


y_1
y_2
y
x
4
3
2
1
−1
−2
−4
−3
−2
−1
1
2
3
4

CHAPTER 7

LIMITS

In Section 1A, we saw how to go from a position graph to a velocity graph. However, the graphs we were dealing with were piecewise linear, which made it very easy to find the velocities, or the slopes. If the position graphs are not piecewise linear, it is more difficult to find the slope at a given point on the graph.

There is a very nice way of doing this for many functions, but if we're not careful, it will require division by zero! What does being careful entail? It means knowing your limits!

NUMERICAL LIMITS

Suppose you wanted to evaluate the function $f(x) = \frac{x^3-8}{x-2}$ at $x = 2$. Plugging in $x = 2$ into the $f(x)$ formula gives

$$\frac{2^3-8}{2-2} = \frac{0}{0}$$

But there is a problem — we've divided by zero.

(image credit: Jaggery)

Hopefully you didn't actually do the division by zero. What can we do instead? Let's make a table to see what happens when we get close to putting in two, without actually doing it.

x	f(x)
1.5	9.25
1.9	11.41
1.99	11.9401
1.999	11.994
1.9999	11.9994
2.0001	12.0006
2.001	12.006
2.01	12.0601
2.1	12.61
2.5	15.25

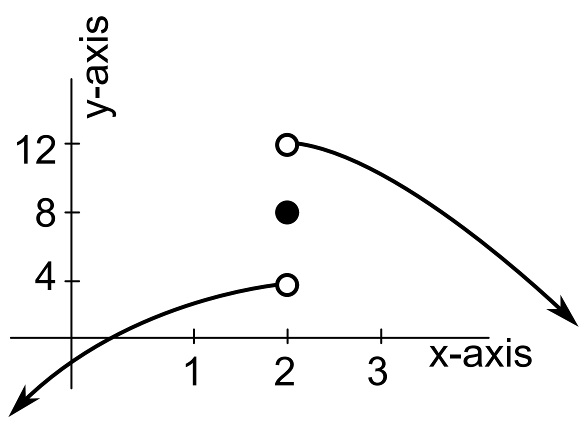
If you look at the table, it looks like $f(x)$ SHOULD be equal to 12 at $x = 2$. We can't plug in $x = 2$ because of division by zero, but it really should be 12 if it has a value. Can we just say it's 12 and call it a day? Well, not quite, since we want to distinguish between functions that are actually equal to 12, and ones that just should be. That's where limits come in.

We say $f(2)$ is undefined, but we can write $\lim_{x\to 2} f(x) = 12$, which we read as "the limit of f of x as x approaches two is twelve". That example is the idea behind a limit.

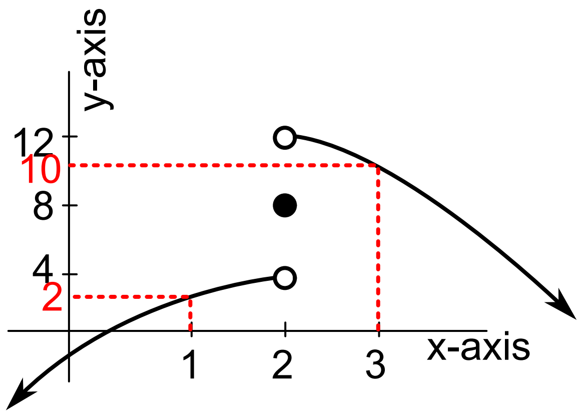
More technically, a *limit* is a value L the y-value of a function $f(x)$ approaches as the x-value approaches a certain value a, either from the right, the left or both. Notationally, $\lim_{x\to a^-} f(x) = L$ is the left hand limit, $\lim_{x\to a^+} f(x) = L$ is the right hand limit, and $\lim_{x\to a} f(x) = L$ is the two-sided limit. Let's look at some pictures to make this more intuitive.

GRAPHICAL LIMITS

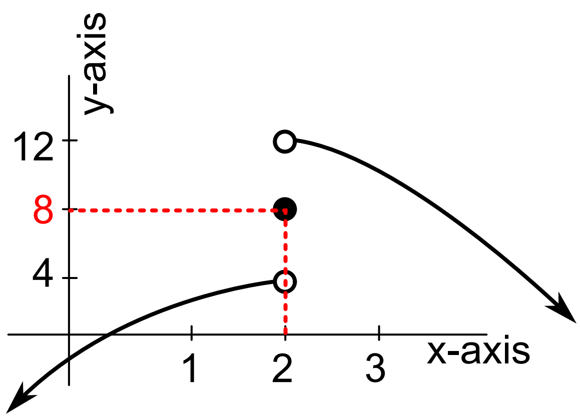
Consider the following $f(x)$.



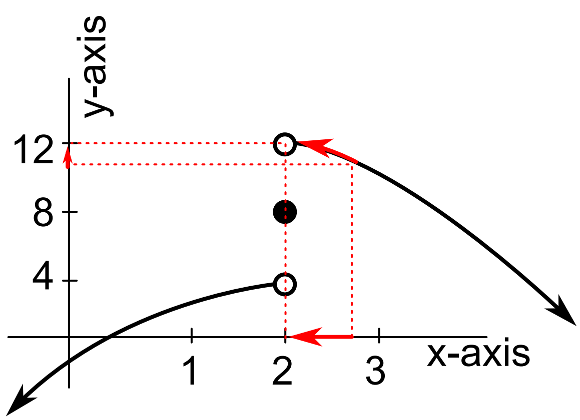
To find the value of a function, recall that you look at how high that function is at a given x value. For example, $f(1)$ is about 2, and $f(3)$ is about 10.



What's happening at $x = 2$, or $f(2)$? The filled-in circle shows the function value. The white circle indicates that value is not part of the function. So $f(2) = 8$.



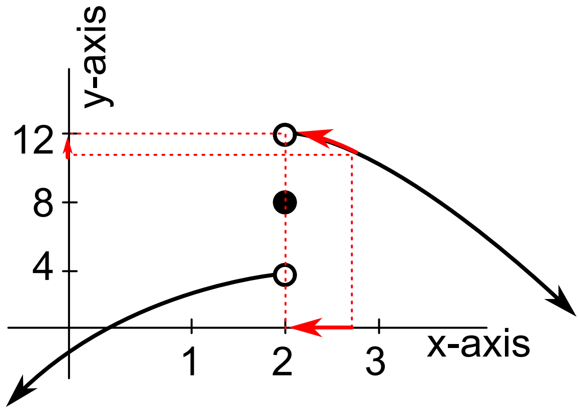
However, there is something funny going on at $x = 2$. Namely, the function seems to "jump from 4, stop at 8 momentarily, then finally jump to 12 and continue. This is called a *discontinuity*, and is usually a bad thing, or at least something that can be hard to deal with. This is where limits can be helpful. The notation $\lim_{x \to 2^-} f(x)$ indicates the y-value of the function as x approaches the value of 2 from the left. Here is the picture:



In other words, $\lim_{x \to 2^-} f(x)$ is the value that $f(x)$ should be at 2, if you were approaching from the left. In this case, $f(x)$ should be 4 if everything were right in the world. Therefore,

$\lim_{x\to 2^-} f(x) = 4$. The limit allows us to fill in what the function should be, even though it isn't the case.

The right handed limit $\lim_{x\to 2^+} f(x)$ is the same way, but it approaches from the right.



Here, the function approaches 12 as we approach 2 from the right, and therefore we write $\lim_{x\to 2^+} = 12$.

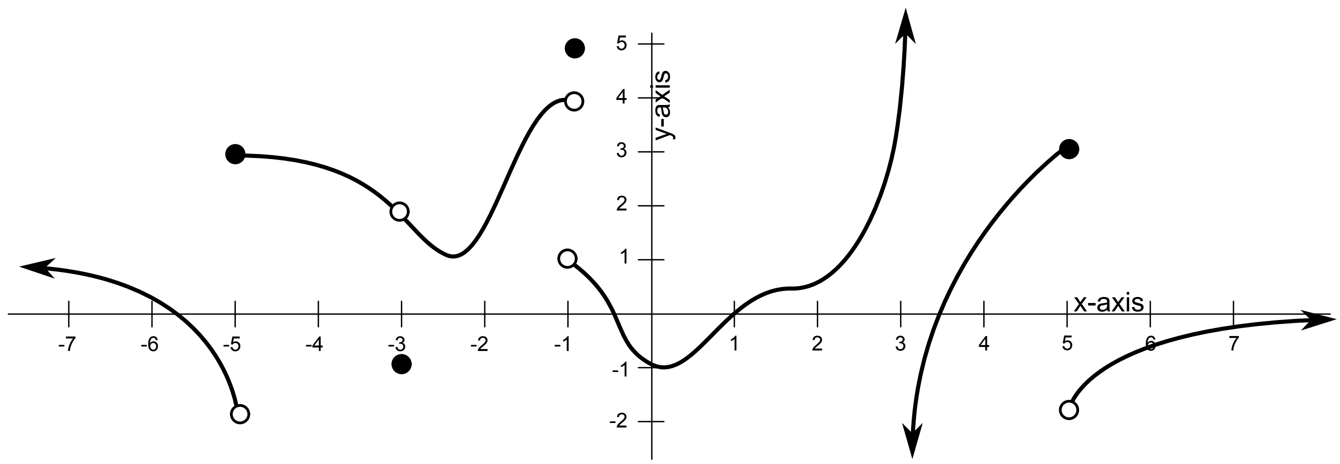
A two-sided limit exists if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$. In other words, if the left and right limits are the same. The notation for this is $\lim_{x\to a} f(x)$. Since the left and right limits are different, so we just write "Does Not Exist" or "DNE". So $\lim_{x\to 2} f(x) = \text{DNE}$.

OTHER EXAMPLES

Problem

For each value of a, find $f(a)$, $\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$, $\lim_{x\to a} f(x)$.

1. $a = -5$
2. $a = -3$
3. $a = -1$
4. $a = 1$
5. $a = 3$
6. $a = 5$
7. $a = \infty$
8. $a = -\infty$



1. We are looking at the function at $x = -5$. The actual value here is $f(-5) = 3$. However, the value it should be if we were coming at -5 from the left is -2, so $\lim_{x\to -5^-} f(x) = -2$. Coming from the right, it should be 3, so $\lim_{x\to -5^+} f(x) = 3$. Finally, $\lim_{x\to -5} f(x) = \text{DNE}$, since the left and right limits are not equal.
2. We are looking at the function at $x = -3$. We have $f(-3) = -1$, since that's where the black dot is. But the value SHOULD be 2, whether we approach from the right or left. So $\lim_{x\to -3^-} f(x) = \lim_{x\to -3^+} f(x) = 2$. Since the left and right limits are equal, we have $\lim_{x\to -3} f(x) = 2$.
3. We have $f(-1) = 5$, $\lim_{x\to -1^-} f(x) = 4$, $\lim_{x\to -1^+} f(x) = 1$, and $\lim_{x\to -1} f(x) = \text{DNE}$.
4. Everything is nice and happy — there are no discontinuities here. Therefore $f(1) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = 0$.
5. We have what is called a vertical asymptote, and the function basically goes off infinitely far in both directions. Here, we say $f(3) = \text{DNE}$, since there is no obvious value to make $f(3)$ equal to. We say $\lim_{x\to 3^-} f(x) = \infty$, since it goes up infinitely high (the ∞ symbol means "infinity"). Likewise, $\lim_{x\to 3^+} f(x) = -\infty$, since it goes down infinitely low. Therefore, we can see $\lim_{x\to 3} f(x)$ does not exist.
6. We have $f(5) = 3$, $\lim_{x\to 5^-} f(x) = 3$, $\lim_{x\to 5^+} f(x) = -2$, and $\lim_{x\to 5} f(x) = \text{DNE}$.
7. We have $x = \infty$. What we are asking here is what happens to $f(x)$ as x gets really big. In other words, what happens to the function as x goes to infinity? Well, it looks like perhaps $f(x)$ is just heading towards zero. So we say $\lim_{x\to\infty} f(x) = 0$. There is no such thing as a right hand or two-sided limit in this case, nor does it make sense to talk about $f(\infty)$. So we just leave it as $\lim_{x\to\infty} f(x) = 0$. This is the same thing as a horizontal asymptote.
8. Similarly we are looking at what happens to $f(x)$ when x goes more and more negative. It looks like maybe $f(x)$ is heading towards 1, so we write $\lim_{x\to -\infty} f(x) = 1$.

CHAPTER 8

HOMEWORK: LIMITS

1. Graph each piecewise defined function.

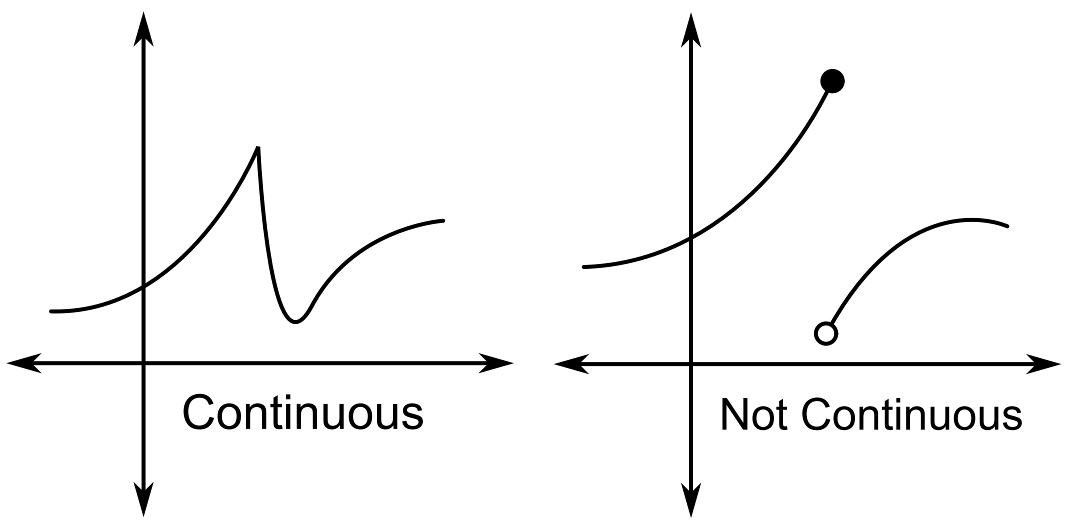
$$\text{a. } f(x) = \begin{cases} 2x+1 & x \leq 1 \\ -x+2 & x > 1 \end{cases}$$

$$\text{b. } g(x) = \begin{cases} \frac{1}{2}x+3 & x \leq -4 \\ -\frac{1}{2}x-1 & x > -4 \end{cases}$$

2. Why is the following not a function?

$$h(x) = \begin{cases} \frac{2}{3}x & x \leq 1 \\ \frac{4}{3}x-5 & x \geq 1 \end{cases}$$

FUNDAMENTALS



3. If $\lim_{x\to a} f(x) = f(a)$, then we say that $f(x)$ is *continuous* at point a. In general, there is a simpler way to think about continuity: if a section of a graph of a function can be drawn without lifting your pencil, then that part of the function is continuous. If you need to lift your pencil at some point, that point is called a *discontinuity*. If the whole function is continuous everywhere, then the function itself is called continuous.

   For each function below, label it as continuous or not continuous. If it is not continuous, list at least one discontinuity.

a. $f(x) = x^2$.
Continuous
ans

b. The piecewise linear function $g(x)$ defined by

$$g(x) = \begin{cases} -x & x \leq 2 \\ 2x & x > 2 \end{cases}$$

Discontinuous at $x = 2$.
ans

c. The piecewise linear function $g(x)$ defined by

$$g(x) = \begin{cases} x+1 & x \leq 3 \\ 2x-2 & x > 3 \end{cases}$$

Continuous
ans

d. The piecewise linear function $g(x)$ defined by

$$g(x) = \begin{cases} 3x+11 & x \leq -5 \\ 0 & x > -5 \end{cases}$$

Discontinuity at $x = -5$
ans

e. Your age in years as a whole number as a function of time.
Discontinuous at every birthday
ans

f. The height of a tennis ball as a function of time.
Continuous
ans

4. In this exercise, we will compute $\lim_{x\to 0} \frac{e^x-1}{x}$ using a calculator.

a. Fill in the following table of values.

x	$\frac{e^x-1}{x}$
-0.1	
-0.01	
-0.001	
0.001	
0.01	
0.1	

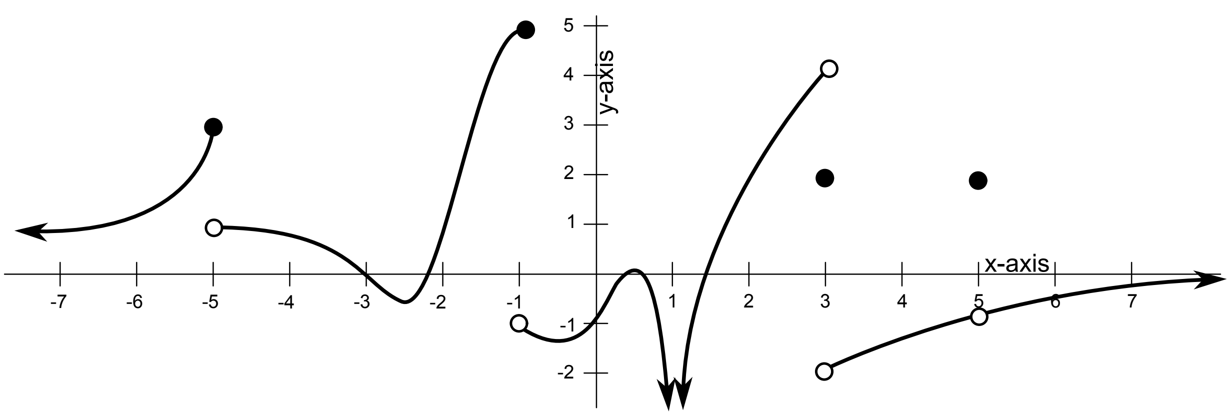
b. What does $\lim_{x\to 0} \frac{e^x-1}{x}$ seem to equal?
1
ans

c. Use a graphing calculator or computer to graph $y = \frac{e^x-1}{x}$. Looking at the graph, what does it look like y is equal to when $x = 0$?
Loooooks like 1.
ans

5. Compute the following limits using a calculator like in problem (4). In each case, sketch a graph and jot down a table of values.

   a. $\lim_{h\to 0^+} 1 - 2^{-1/h}$
      1
      ans
   b. $\lim_{h\to 3} \frac{x^2-9}{x-3}$
      6
      ans

6. Watch the following KhanAcademy video link: One Sided Limits from Graphs
7. Find the following values given the graph of $f(x)$ below.



   a. $\lim_{a\to -5^+} f(a)$
      1
      ans
   b. $\lim_{b\to -5^-} f(b)$
      3
      ans
   c. $f(-5)$
      3
      ans
   d. $\lim_{d\to -3^+} f(d)$
      0
      ans
   e. $\lim_{e\to -3^-} f(e)$
      0
      ans
   f. $f(-3)$
      0
      ans
   g. $\lim_{g\to -1^+} f(g)$
      -1
      ans
   h. $\lim_{h\to -1^-} f(h)$
      5

ans

i. $f(-1)$
5
ans

j. $\lim_{j \to 1^+} f(j)$
DNE (or $-\infty$)
ans

k. $\lim_{k \to 1^-} f(k)$
DNE (or $-\infty$)
ans

l. $f(1)$
DNE
ans

m. $\lim_{m \to 3^+} f(m)$
-2
ans

n. $\lim_{n \to 3^-} f(n)$
4
ans

o. $f(3)$
2
ans

p. $\lim_{p \to 5^+} f(p)$
-1
ans

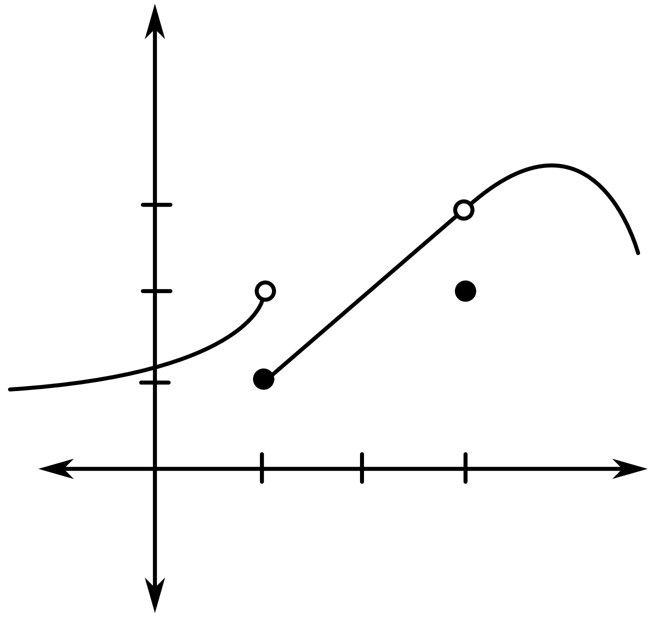
q. $\lim_{q \to 5^-} f(q)$
-1
ans

r. $f(5)$
2
ans

s. $\lim_{x \to \infty} f(x)$
0
ans

t. $\lim_{x \to -\infty} f(x)$
1
ans

8. From $f(x)$ from Problem 7, list whether $f(x)$ is continuous or not continuous at the following values of x: -5, -3, -1, 1, 3, 5.
Only continuous at $x = -3$
ans
9. Watch the following KhanAcademy video link:
Two-sided limit from graph
10. Compute (as close as you can tell from the graph) the limit or function value in each case. If the limit does not exist, explain why.



a. $\lim_{x\to 2} f(x)$
   2
   ans
b. $f(2)$
   2
   ans
c. $\lim_{x\to 1} f(x)$
   Does not exist
   ans
d. $f(1)$
   1
   ans
e. $\lim_{x\to 3} f(x)$
   3
   ans
f. $f(3)$
   2
   ans

11. Is it possible that $f(2) = 3$ but $\lim_{x\to 2} f(x) = -3$? If so, sketch a picture of the graph of such an $f(x)$ If not, explain why not.
    Yes, it is possible.
    ans
12. Is it possible that $\lim_{x\to 2^+} f(x) = 3$ but $\lim_{x\to 2} f(x) = -3$? If so, sketch a picture of the graph of such an $f(x)$. If not, explain why not.
13. Suppose $\lim_{x\to -5} f(x) = -2$.

    a. What does $f(x)$ approach as x approaches -5?
       -2
       ans
    b. What is $f(-5)$?

Unknown
ans

c. What does x approach if $f(x)$ is approaching -2?
-5 but also perhaps other values
ans

14. Suppose $\lim_{x\to 2} f(x) = -3$.

    a. Estimate $f(1.99)$.
    It's probably close to -3
    ans
    b. Is is possible $f(1.99) = -42$? Why or why not?
    This is possible but unlikely. There is nothing in the definition of limit that says $f(1.99)$ can't equal -42 if the limit as $x \to 2$ is -3.
    ans

15. For each part, sketch a graph of what $f(x)$ might look like. Each problem is separate and will probably require a different graph.

    a. $\lim_{x\to 3} f(x) = 4$ and $f(3) = 4$.
    b. $\lim_{x\to 3} f(x)$ does not exist.
    c. $\lim_{x\to 3^+} f(x) = -2$ and $\lim_{x\to 3^-} f(x) = 5$.
    d. $\lim_{x\to -2} f(x) = 3$ and $f(-2) = 5$.

CHAPTER 9

ALGEBRA TIPS AND TRICKS PART III (FACTORING)

FACTORING

When factoring an expression like this:

$$x^2 - 8x + 15$$

The goal is to write this like $(x+a)(x+b)$ for some numbers a and b, where a and b could be positive, negative, or zero. Since $(x+a)(x+b) = x^2 + (a+b)x + (ab)$ we see we need $a+b=-8$ and $ab=15$. That way, when you foil it back out, you have $x^2-8x+15$. We see if $a=-3$ and $b=-5$, this works for both $a+b=-8$ and $ab=15$. Thus,

$$x^2 - 8x + 15 = (x-3)(x-5)$$

Let's do a couple more examples.

- Problem Factor x^2+3x+2.

In this case we want $a+b=3$ and $ab=2$. $a=1$ and $b=2$ works, so $x^2+3x+2=(x+1)(x+2)$.

- Problem Factor $x^2+5x-84$.

This is a bit harder because the numbers are bigger, but we can still do it. We want $a+b=5$, and $ab=-84$. We can see that 84 is 12 times 7. So if we have $a=12$ and $b=-7$, then $a+b=5$ and $ab=-84$. Hence $x^2+5x-84=(x+12)(x-7)$.

- Problem Factor x^2-64.

In this case, we want $a+b=0$ and $ab=-64$. But notice that this means $a=-b$, and hence $-a^2=-64$, which means $a^2=64$. That means $a=8$, so $b=-8$ (or vice versa). Hence $x^2-64=(x+8)(x-8)$.

Chapter 10

ALGEBRAIC LIMITS

A lot of times you don't need to look at a graph or make a table to find a limit. Consider the following:

Problem Find $\lim_{x\to 3} 2x + 1$.

For this problem, we could make a table, or look at a graph, but $2x + 1$ is such a nice function that there really isn't any point in doing all that. Everything is happy at $x = 3$, so we can just plug in that value.

$$\lim_{x\to 3} 2x + 1 = 2(3) + 1 = 7.$$

In fact, any polynomial, logarithmic, or exponential function that you might run into is what is called *continuous*. Continuous means that the limit is what you get if you just plug the value into the function. So why bother with limits? Well, there are some functions that are not continuous, such as rational functions. And rational functions are exactly what crop up when taking derivatives. Consider this example:

Problem Find $\lim_{x\to 3} \frac{x^2-2x-3}{x-3}$.

Look at what happens when you plug in $x = 3$.

$$\begin{aligned}\lim_{x\to 3} \frac{x^2 - 2x - 3}{x - 3} &=^? \frac{(3)^2 - 2(3) - 3}{(3) - 3} \\ &= \frac{0}{0}.\end{aligned}$$

This is the same problem we saw in the numerical limits section. The fraction $\frac{0}{0}$ is called the *indeterminate case*. Here is where a graph or a table might be useful.

x	f(x)
2.5	3.5
2.9	3.9
2.99	3.99
2.999	3.999
2.9999	3.9999
3.0001	4.0001
3.001	4.001
3.01	4.01
3.1	4.1
3.5	4.5

Looks like the limit should be equal to 4.

But there is algebraic way. Just remember this: factor the top, and cancel like terms. Let's see it in action.

$$\begin{aligned}\lim_{x\to 3}\frac{x^2-2x-3}{x-3} &= \lim_{x\to 3}\frac{(x-3)(x+1)}{x-3}\\ &= \lim_{x\to 3}\frac{\cancel{(x-3)}(x+1)}{\cancel{x-3}}\\ &= \lim_{x\to 3} x+1\\ &= (3)+1\\ &= 4\end{aligned}$$

Notice though that it's not right to say $\frac{(x-3)(x+1)}{x-3} = x+1$ always, since they are not equal when $x=3$. At $x=3$, $\frac{(x-3)(x+1)}{x-3}$ is not defined, and $x+1$ is. However, if you're inside a limit as $x\to 3$, then x is not equal to 3, it is just approaching 3. Hence it is okay to cancel those terms.

When dealing with rational expressions, sometimes you can't cancel terms but can still find the value of the limit. For example,

Problem Find $\lim_{x\to -2}\frac{x^2+2}{x-2}$,

The numerator does not factor. So there isn't anything we can cancel. But if we just plug in $x=-2$, we see

$$\lim_{x\to -2} \frac{x^2+2}{x-2} = \frac{(-2)^2+2}{(-2)-2} = \frac{6}{-4} = -\frac{3}{2}.$$

In this example, the top is approaching 6, and the bottom -4. Since there is no division by zero here, the limit value is just $-\frac{3}{2}$. Easy peasy!

If you plug in a value to take a limit and you get a nonzero number divided by zero, you can just say the limit does not exist. For example,

Problem $\lim_{x\to -3} \frac{x}{x+3}$,

we see

$$\lim_{x\to -3} \frac{x}{x+3} = \frac{-3}{(-3)+3} = \frac{-3}{0}.$$

Here, the denominator is approaching zero, while the numerator is holding steady at -3. The result of this is dividing by smaller and smaller numbers, which means the value is getting bigger and bigger to infinity. Thus, the limit does not exist. Check out the table for this problem.

x	$f(x)$	**Simplified**
-3.5	$\frac{-3.5}{-0.5}$	$= 7$
-3.1	$\frac{-3.1}{-0.1}$	$= 31$
-3.01	$\frac{-3.01}{-0.01}$	$= 301$
-3.001	$\frac{-3.001}{-0.001}$	$= 3001$
-2.999	$\frac{-2.999}{0.001}$	$= -2999$
-2.99	$\frac{-2.99}{0.01}$	$= -299$
-2.9	$\frac{-2.9}{0.1}$	$= -29$

As you can see, $f(x)$ is really large when x is close to -3. In short: $\frac{0}{0}$ is the indeterminate case where factoring and cancelling is a good idea. Anything else divided by zero is easy to determine: does not exist.

Example **Algebraic limits**
Find the following limit values algebraically in each case.

1. Problem $\lim_{x \to 4} \frac{7x-4}{2x-4}$

Here, we just need to plug in $x = 4$ and we have $\frac{7(4)-4}{2(4)-4} = \frac{24}{4} = \boxed{6}$. Since we didn't divide by zero, this is continuous at this point, so we can just plug in the value.

2. Problem $\lim_{x \to 2} \frac{x^2-4}{x-2}$

Classic example where we factor the top and cancel.

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x \to 2} x + 2 = \boxed{4}$$

3. Problem $\lim_{x \to 3} \frac{x+3}{x-3}$

In this example, we plug in $x = 3$ and have $\frac{6}{0}$. Since we are dividing by zero, this $\boxed{\text{limit does not exist}}$.

4. Problem $\lim_{x \to -1} \frac{x^2+6x+5}{x+1}$

Another example of factoring and cancelling.

$$\lim_{x \to -1} \frac{x^2+6x+5}{x+1} = \lim_{x \to -1} \frac{(x+1)(x+5)}{x+1} = \lim_{x \to -1} x + 5 = \boxed{4}$$

5. Problem $\lim_{x \to 1} \frac{(x-1)^3}{x-1}$

Another example of factoring and cancelling, but this time the top is already factored!

$$\lim_{x \to 1} \frac{(x-1)^3}{x-1} = (x-1)^2 = (1-1)^2 = 0^2 = \boxed{0}.$$

Chapter 11

HOMEWORK: ALGEBRAIC LIMITS

1. Factor the following polynomials.

    a. $x^2 - 5x + 6$
    $(x-2)(x-3)$
    ans

    b. $x^2 + 2x - 63$
    $(x-7)(x+9)$
    ans

    c. $x^2 - 4xh + 4h^2$
    $(x-2h)^2$
    ans

2. If you're doing a limit with a continuous function, like $\lim_{x\to 3} 7x - 1$, how can you quickly solve this limit problem?
Plug it in!
ans

3. Compute the following limits algebraically.

    a. $\lim_{x\to 5} 2x^2 - 7x$
    15
    ans

    b. $\lim_{x\to 2} \frac{(x-5)(x-2)}{x-2}$
    -3
    ans

    c. $\lim_{x\to 3} \frac{x^2-6x+9}{x-3}$
    0
    ans

    d. $\lim_{x\to -4} \frac{x^2+3x-4}{x+4}$
    -5
    ans

    e. $\lim_{x\to 2} \frac{x^3-6x^2+8x}{x-2}$
    -4
    ans

    f. $\lim_{x\to w} x^2 + 5xw - w^2$
    $5w^2$

ans

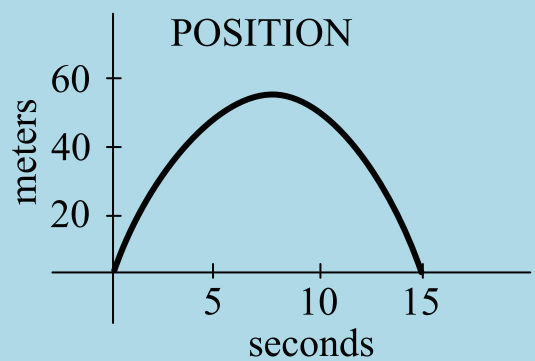
g. $\lim_{x \to h} \frac{x^2 - h^2}{x - h}$
$2h$
ans

Chapter 12

Instantaneous Velocity

What can we say about velocity if the position graph is a curve?

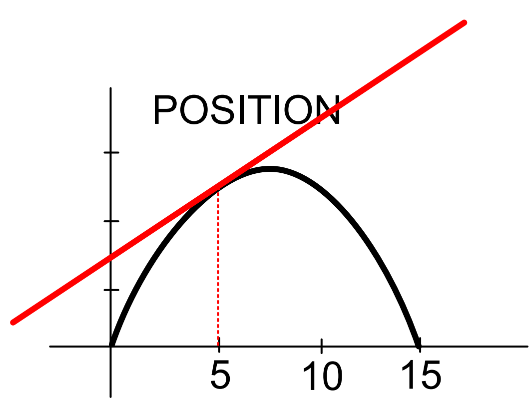
Problem For example, if $p(t) = 15t - t^2$, we'd get a graph that looks like this.



What is the velocity at $t = 5$? Seriously, what is it? I want to know.

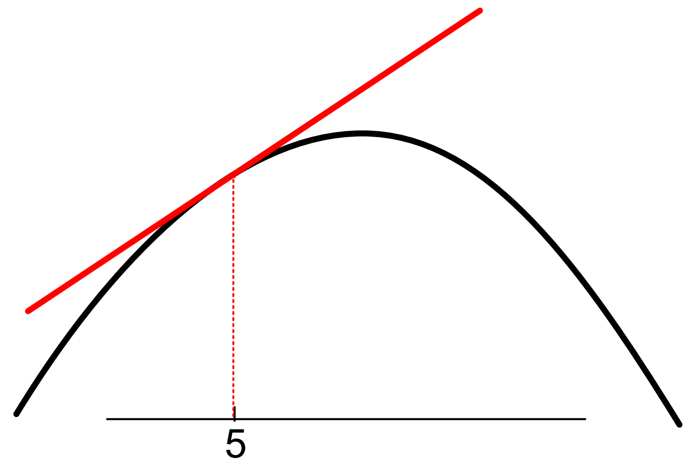
Well, a first stab might be that it is the slope, since that is what we said before. But generally slope is only applied to lines, not curves like this. How could we find slope for a curve?

Well, recall that slope is just how steeply something is increasing. It turns out, we can take a line that is increasing with the same steepness as the curve at a given point, and measure the slope of the line.

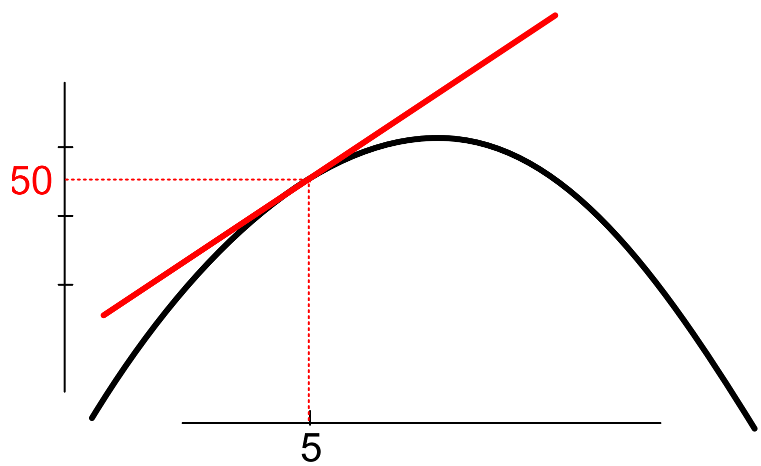


Here, the red line is the same steepness as the curve at $t = 5$. This is called a *tangent line*. Note that often the tangent line only touches the curve once. Since it is a line, we can measure the slope, and this should represent the velocity at $t = 5$. But since it touches **one** time, we don't have **two** points to compute the slope. This may seem like a minor problem, but to find the exact slope takes one of the major insights in calculus: **we need to develop a process to get closer and closer, and then use a limit to find the exact value**. This might seem like a lot of work but it is worth it, as it demonstrates the power of calculus.

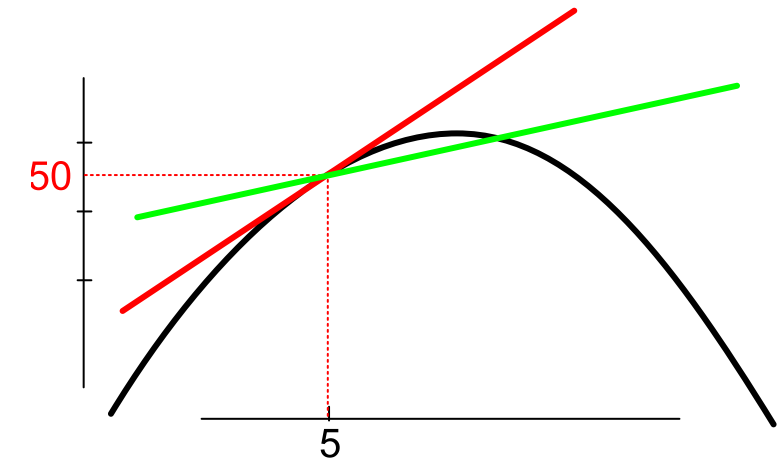
Let's zoom in a bit on the graph:



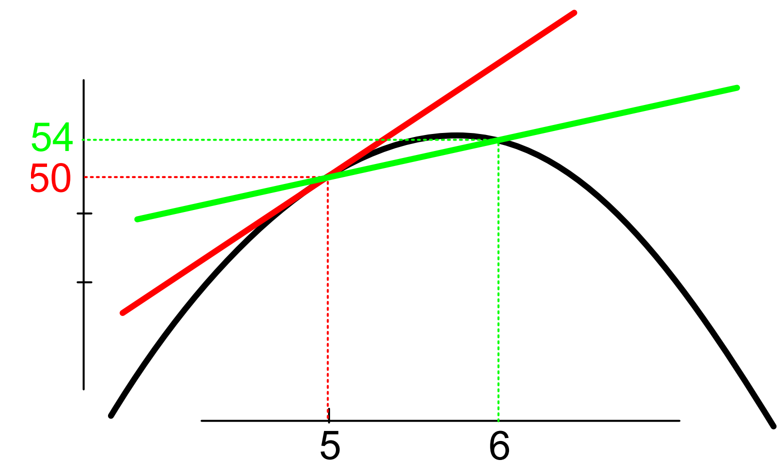
Okay, now we can start to see some things. First, notice we know the y-value or function value at the point $t = 5$. Why? Since the function $p(t) = 15t - t^2$ is giving the position, we can plug in $t = 5$. We see $p(5) = 15(5) - (5)^2 = 50$.



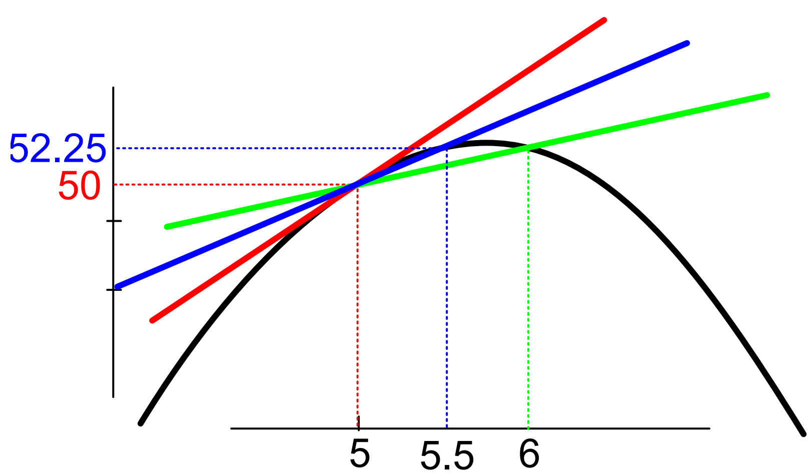
But this doesn't give us the slope. To find a slope, you need two points — then you can use the rise over run formula. But we only have one point. So instead let's look at a line that does have two intersection points, but is not quite the line we want.



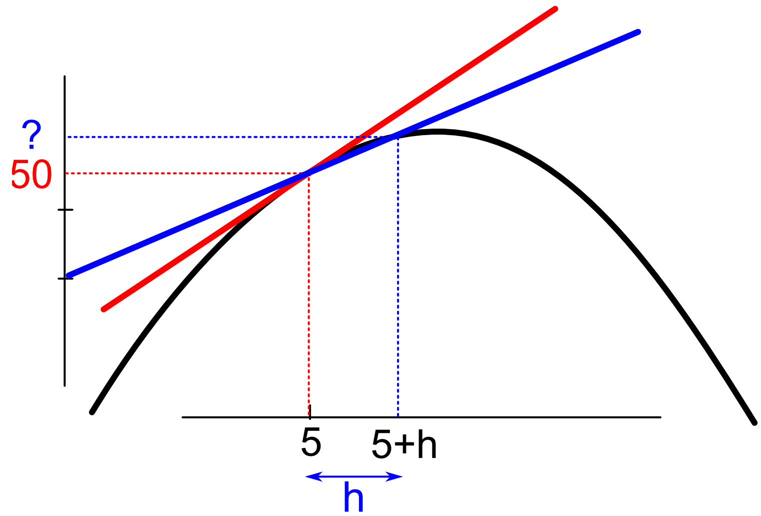
Let's say this new green line hits the curve at $t = 6$. A line like this green one that hits in two locations is called a **secant line**. What is the y-value at $t = 6$? Well, it's $p(6) = 15(6) - (6)^2 = 54$.



Now we can find the slope of the green line. It's $\frac{54-50}{6-5} = \frac{4}{1} = 4$, since that is what rise over run tells us. But again, it isn't quite the line we want. Instead, we could choose a blue line that's even closer.



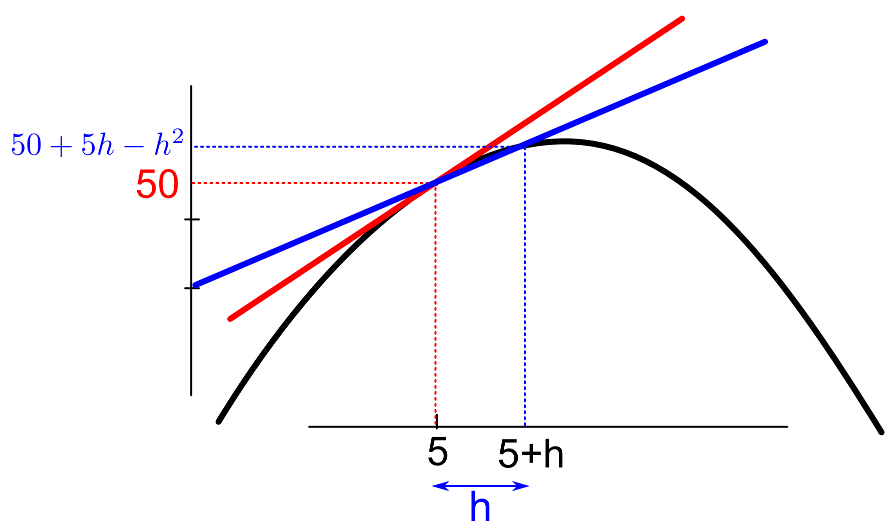
Now the slope of the blue line is $\frac{52.25-50}{5.5-5} = \frac{2.25}{0.5} = 4.5$. Still not there. But we can get closer and closer and closer. Instead of repeating this calculation every time, let us use variables. So instead of the blue line crossing at $t = 5.5$, let's just say it crosses at $t = 5 + h$, or h to the right of where the red line crosses.



What is the blue question mark? Well, we just plug $t = 5 + h$ in to $p(t) = 15t - t^2$. We see

$$\begin{aligned} p(5+h) &= 15(5+h) - (5+h)^2 \\ &= 75 + 15h - (25 + 10h + h^2) \\ &= 75 + 15h - 25 - 10h - h^2 \\ &= 50 + 5h - h^2 \end{aligned}$$

There you go.



Now the slope is rise over run, and we have

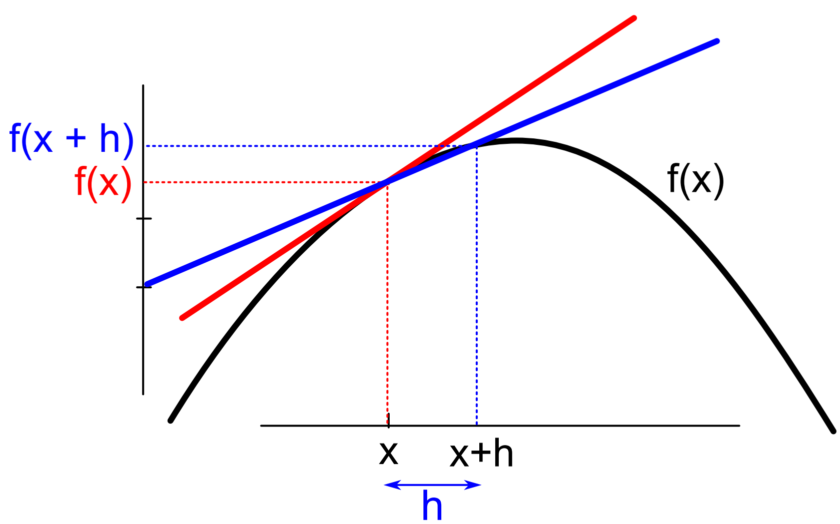
$$\text{slope} = \frac{50 + 5h - h^2 - 50}{5 + h - 5}$$
$$= \frac{5h - h^2}{h}$$

Okay, what we are really interested in is when h gets really, really small. That's when the blue line slope will equal the red line slope, and the red line slope is what we want. But we can't plug in $h = 0$, since that would give a division by zero explosion. So instead, we can use a limit!

$$\text{slope} = \lim_{h\to 0} \frac{5h - h^2}{h}$$
$$= \lim_{h\to 0} \frac{h(5 - h)}{h}$$
$$= \lim_{h\to 0} (5 - h)$$
$$= 5 - (0)$$
$$= 5$$

And there you have it. The slope of the red line is 5!

Now, suppose instead of $t = 5$, we were interested in the instantaneous velocity at $t = x$. And suppose instead of $15t^2 - t^2$, we had some other function describing position, which we call $f(t)$. What would the picture look like then? Well, it would look very similar to the last picture.



And this is the picture that gives us the definition of the derivative. What is the blue slope? It's

$$\text{Blue Slope} = \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

So how do we find the red slope? We just take the limit as the blue line approaches the red — that is, see what happens as h goes to zero.

Definition of Derivative:
$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Here is the notation for the derivative: $f'(x)$, $\frac{df}{dx}$, or $\frac{d}{dx} f(x)$.

Chapter 13

HOMEWORK: INSTANTANEOUS VELOCITY

1. The position of a falling object follows the equation $f(t) = -16t^2 + 64$ from $t = 0$ to $t = 2$.

    a. Verify that the points $(1, 48)$ and $(2, 0)$ are on the curve by computing $f(1)$ and $f(2)$ and verifying you get 48 and 0.
    This seems to work.
    ans
    b. Compute the slope of the line going through $(1, 48)$ and $(2, 0)$.
    -48
    ans
    c. Verify that the points $(1, 48)$ and $(1 + h, -16h^2 - 32h + 48)$ lie on the curve. You've already done $(1, 48)$, so now just simplify $f(1 + h)$ and verify you get $-16h^2 - 32h + 48$.
    This seems to work.
    ans
    d. Compute the slope of the line through $(1, 48)$ and $(1 + h, -16h^2 - 32h + 48)$ (hint: you should get $-16h - 32$!)

2. The position of a falling object follows the equation $f(t) = -5t^2 + 45$ from $t = 0$ to $t = 3$.

    a. Verify that the points $(2, 25)$ and $(3, 0)$ line on the curve, and compute the slope through these two points.
    The slope is -25
    ans
    b. Verify that the points $(2, 25)$ and $(2 + h, -5h^2 - 20h + 25)$ lie on the curve, and compute the slope of the line through these two points.

    The slope is $-5h - 20$.
    ans

3. Let $g(t) = -10t^2 + 2000$ represent a population of goats, where $g(t)$ is measured in goats and t is measured in years. Suppose t only works on the range from 0 to 10. This population is stabilizing during this period.

    a. Sketch a graph of $g(t)$.
    b. Find the slope of the secant line hitting $g(t)$ at $t = 2$ and $t = 3$.

-50 goats per year
ans

c. The slope of the secant line hitting $g(t)$ at $t = 2$ and $t = 2 + h$.
$-40 - 10h$ goats per year
ans
d. What is $\lim_{h \to 0}$ for your answer in part (c)?
-40 goats per year.
ans
e. How quickly is the goat population growing at $t = 2$?
-40 goats per year.
ans

Chapter 14

ALGEBRA TIPS AND TRICKS IV (TIPS FOR DEALING WITH FRACTIONS)

A QUICK FRACTION HINT

A couple of ideas while working with fractions. Note that if you distribute a number times a fraction, you multiply on top:

$$x\left(\tfrac{1}{2}+y\right)=\tfrac{x}{2}+xy$$

The reason is when we multiply fractions, we multiply straight across, and we can always think of x as $\frac{x}{1}$. Hence $x\cdot\frac{1}{2}=\frac{x}{1}\cdot\frac{1}{2}=\frac{x}{2}$.

COMPLEX FRACTIONS

If you have "fractions within fractions", this calls out to be simplified. One way to do it is to multiply top and bottom of the outer fraction by the same number so that the inner fractions go away. For example,

Problem

$$\frac{\dfrac{1}{3}-\dfrac{\sqrt{3}}{3}}{\dfrac{1}{9}}$$

If we multiply top and bottom by 9 in the following example, that gets rid of the 3 and the 9 denominators:

$$\begin{aligned}\left(\frac{\dfrac{1}{3}-\dfrac{\sqrt{3}}{3}}{\dfrac{1}{9}}\right)\cdot\frac{9}{9} &= \frac{9\cdot\dfrac{1}{3}-9\cdot\dfrac{\sqrt{3}}{3}}{9\cdot\dfrac{1}{9}}\\ &=\frac{3-3\sqrt{3}}{1}\\ &=3-3\sqrt{3}\end{aligned}$$

You can also multiply by the reciprocal instead of dividing. Like this:

$$\begin{aligned}\left(\frac{\frac{1}{3}-\frac{\sqrt{3}}{3}}{\frac{1}{9}}\right) &= \left(\frac{1}{3}-\frac{\sqrt{3}}{3}\right)\cdot\frac{9}{1} \\ &= 9\cdot\frac{1}{3}-9\cdot\frac{\sqrt{3}}{3} \\ &= 3-3\sqrt{3}\end{aligned}$$

The same thing happens with variables. Consider this problem:

Problem

$$\frac{\frac{1}{x}+\frac{1}{y}}{x}$$

We can simplify this by multiplying by xy to get rid of the x and y denominators on top.

$$\begin{aligned}\frac{\frac{1}{x}+\frac{1}{y}}{x} &= \frac{\frac{1}{x}+\frac{1}{y}}{x}\cdot\frac{xy}{xy} \\ &= \frac{\frac{xy}{x}+\frac{xy}{y}}{xxy} \\ &= \frac{y+x}{x^2y}\end{aligned}$$

Here's one more example.

Problem Simplify $\dfrac{\frac{1}{x+1}-\frac{1}{x-1}}{\frac{1}{x}}$.

To simplify this one, we need to clear all the denominators by multiply by x, $(x+1)$ and $(x-1)$. Not easy, but we can do it.

$$
\begin{aligned}
\frac{\dfrac{1}{x+1} - \dfrac{1}{x-1}}{\dfrac{1}{x}} \cdot \frac{x(x-1)(x+1)}{x(x-1)(x+1)} &= \frac{\dfrac{x(x+1)(x-1)}{x+1} - \dfrac{x(x+1)(x-1)}{x-1}}{\dfrac{x(x+1)(x-1)}{x}} \\
&= \frac{x(x-1) - x(x+1)}{(x-1)(x+1)} \\
&= \frac{x^2 - x - (x^2 + x)}{x^2 - 1} \\
&= \frac{x^2 - x - x^2 - x}{x^2 - 1} \\
&= \frac{-\,2x}{x^2 - 1}
\end{aligned}
$$

CHAPTER 15

DEFINITION OF DERIVATIVE EXAMPLES

In the last section, we saw the instantaneous rate of change, or derivative, of a function $f(x)$ at a point x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

Example **Definition of Derivative 1**

Problem Find the derivative of the function $f(x) = 3x + 5$ using the definition of the derivative.

To use this in the formula $f'(x) = \frac{f(x+h)-f(x)}{h}$, first we need to replace the $f(x+h)$ part of the formula. This is the same as $f(x)$ which is $3x+5$, except we replace x with that $(x+h)$ in parantheses. Like the following. The colors are only to highlight the substitution of $f(x+h)$ and $f(x)$. We'll drop the colors as soon as we need to combine expressions.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{3(x+h) + 5 - 3x + 5}{h} \end{aligned}$$

Now we continue to simplify and find the answer.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{3(x+h)+5 - 3x+5}{h} \\
&= \lim_{h\to 0} \frac{3x+3h+5-3x-5}{h} \\
&= \lim_{h\to 0} \frac{3h}{h} \\
&= \lim_{h\to 0} 3 \\
&= \boxed{3}
\end{aligned}
$$

Here, we have $f'(x) = 3$. That makes sense if you think about it: $3x + 5$ is a line with slope 3!

Example **Definition of Derivative 2**

Problem Find the derivative of $f(x) = x^2$ using the definition.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} \frac{h(2x+h)}{h} \\
&= \lim_{h\to 0} 2x + h \\
&= 2x + (0) \\
&= \boxed{2x}
\end{aligned}$$

So what does this mean? Well, this means we double x to find the slope of the tangent line of $f(x) = x^2$. So at $x = 3$, the slope is 6, and at $x = 1.2$, the slope is 2.4. ETC.

CHAPTER 16

PROJECT: HARD DEFINITION OF DERIVATIVE PROBLEMS

Purpose of the project: Struggle through a difficult but important calculus problem. Each of the following is a difficult definition of the derivative problem. Your group will be assigned one of the following, and then you can present the solution to the class. In each case, the "stuff in the example box" is not your problem, but look at it and hopefully it will help with your problem.

1. Let $f(x) = x^4$. Using the definition of the derivative, find $f'(x)$.

   Example Simplify $(x+1)^4$.We can do this problem by splitting it up into $(x+1)^2(x+1)^2$. We know $(x+1)^2 = x^2+2x+1$ — that means

   $$(x+1)^4 = (x^2+2x+1)(x^2+2x+1)$$

   To solve from here, we need to multiply every term of (x^2+2x+1) by x^2, then every term by $2x$, then every term by 1, and add it all up. Here we go:

   $$\begin{aligned}(x+1)^4 &= (x^2+2x+1)(x^2+2x+1)\\ &= (x^2+2x+1)x^2 + (x^2+2x+1)2x + (x^2+2x+1)1\\ &= x^4+2x^3+x^2+2x^3+4x^2+2x+x^2+2x+1\\ &= x^4+4x^3+6x^2+4x+1.\end{aligned}$$

2. Let $f(x) = \sqrt{x}$. Using the definition of the derivative, find $f'(x)$ (which can also be written $\frac{d}{dx}f(x)$ or $\frac{df}{dx}$).

   Example Rationalize the numerator of $\frac{\sqrt{x+1}-\sqrt{x}}{x}$.

   To "rationalize the numerator", the trick is to multiply top and bottom by what is known as the conjugate: it is the same as the numerator, only the sign is flipped so that subtraction becomes addition or vice versa. In this case, the conjugate is $\sqrt{x+1}+\sqrt{x}$. We see

$$\begin{aligned}
\frac{\sqrt{x+1}-\sqrt{x}}{x} &= \frac{(\sqrt{x+1}-\sqrt{x})}{(x)} \cdot \frac{(\sqrt{x+1}+\sqrt{x})}{(\sqrt{x+1}+\sqrt{x})} \\
&= \frac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{(x)(\sqrt{x+1}+\sqrt{x})} \\
&= \frac{(\sqrt{x+1})^2-(\sqrt{x})^2}{x(\sqrt{x+1}+\sqrt{x})} \\
&= \frac{x+1-x}{x(\sqrt{x+1}+\sqrt{x})} \\
&= \frac{1}{x(\sqrt{x+1}+\sqrt{x})}
\end{aligned}$$

The numerator is now rationalized.

3. Let $f(x) = \sin(x)$. Using the definition of the derivative, find $f'(x)$ (which can also be written $\frac{d}{dx}f(x)$ or $\frac{df}{dx}$). There are three facts we need to compute this derivative:
   - $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$
   - $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$
   - $\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$

Example Find $\lim_{h\to 0} \dfrac{\sin(y+h)-\sin(y)\cos(h)}{h}$.

In this example, we use the first fact listed above to write $\sin(y+h) = \sin(y)\cos(h) + \cos(y)\sin(h)$. We have the original problem is equal to

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{\sin(y)\cos(h) + \cos(y)\sin(h) - \sin(y)\cos(h)}{h} \\
&= \lim_{h\to 0} \frac{\cos(y)\sin(h)}{h} \\
&= \lim_{h\to 0} \frac{\sin(h)}{h} \cdot \cos(y) \\
&= 1 \cdot \cos(y) \\
&= \cos(y)
\end{aligned}
$$

Notice towards the end we used $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$.

4. Let $f(x) = e^x$. Using the definition of the derivative, find $f'(x)$ (which can also be written $\frac{d}{dx} f(x)$ or $\frac{df}{dx}$). Use the limit we found on the homework yesterday: $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$.

   Example Helpful example: Simplify $\frac{e^{x+1}+e^1}{e^x+1}$.

   Recall that $e^{x+1} = e^x e^1$. Hence this becomes

$$
\begin{aligned}
\frac{e^{x+1}+e^1}{e^x+1} &= \frac{e^x e^1 + e^1}{e^x+1} \\
&= \frac{e^1(e^x+1)}{e^x+1} \\
&= e^1 = e
\end{aligned}
$$

5. Let $f(x) = \frac{1}{x}$. Using the definition of the derivative, find $f'(x)$ (which can also be written $\frac{d}{dx} f(x)$ or $\frac{df}{dx}$).

   Example Simplify $\dfrac{\dfrac{1}{x+1} - \dfrac{1}{x}}{x}$.

   In this example, we want to clear the denominators from this tricky double fraction. To do this, we will multiply by both denominators: $x(x+1)$.

$$\begin{aligned}
\frac{\frac{1}{x+1}-\frac{1}{x}}{x} &= \frac{\left(\frac{1}{x+1}-\frac{1}{x}\right)}{x}\cdot\frac{x(x+1)}{x(x+1)} \\
&= \frac{\left(\frac{1}{x+1}-\frac{1}{x}\right)x(x+1)}{(x)(x)(x+1)} \\
&= \frac{\left(\frac{x(x+1)}{x+1}-\frac{x(x+1)}{x}\right)}{x^2(x+1)} \\
&= \frac{x-(x+1)}{x^2(x+1)} \\
&= \frac{-1}{x^2(x+1)}
\end{aligned}$$

6. Let $f(x) = \ln(x)$. Using the definition of the derivative, find $f'(x)$ (which can also be written $\frac{d}{dx}f(x)$ or $\frac{df}{dx}$). We will need the following facts:
   - $e^x = \lim_{h\to 0}(1+hx)^{1/h}$
   - $\ln(A) - \ln(B) = \ln(A/B)$
   - $n\ln(A) = \ln(A^n)$
   - $\ln(e^x) = x$

Example (Example 1): Rewrite $\frac{\ln(x+5)-\ln(x)}{2}$ as the natural log of a single quantity.

We need to use some log rules to simplify this. First we use $\ln(A) - \ln(B) = \ln(A/B)$
.

$$\frac{\ln(x+5)-\ln(x)}{2} = \frac{\ln\left(\frac{x+5}{x}\right)}{2}$$
$$= \frac{\ln\left(1+\frac{5}{x}\right)}{2}$$

Now we use $n\ln(A) = \ln(A^n)$, thinking of the division by 2 as a multiplication by $1/2$.

$$\frac{\ln\left(1+\frac{5}{x}\right)}{2} = \frac{1}{2}\ln\left(1+\frac{5}{x}\right)$$
$$= \ln\left(\left(1+\frac{5}{x}\right)^{1/2}\right)$$

Example (Example 2): Simplify $\lim_{h\to 0}\left(1+\frac{3h}{k}\right)^{1/h}$.

In this example, we know from our facts above that $e^x = \lim_{h\to 0}(1+hx)^{1/h}$. We also see that

$$\lim_{h\to 0}\left(1+\frac{3h}{k}\right)^{1/h} = \lim_{h\to 0}\left(1+\frac{3h}{k}\right)^{1/h}$$
$$= \lim_{h\to 0}\left(1+h\left(\frac{3}{k}\right)\right)^{1/h}$$

Notice now that $\frac{3}{k}$ is playing the same role as x in the e^x equation. So this simplifies to $e^{3/k}$.

Chapter 17

HOMEWORK: EXAMPLES OF THE DEFINITION OF THE DERIVATIVE

---

1. Simplify each expression involving fractions or rational expressions.

   a. $(x+1)\cdot\left(\frac{1}{3}+\frac{x}{x+1}\right)$
   $= \frac{x+1}{3} + x = \frac{4x+1}{3}$
   ans

   b. $$\frac{\dfrac{1}{3}+1}{1-\dfrac{1}{3}}$$
   2
   ans

   c. $$\frac{x+1}{\dfrac{1}{x}}$$
   x^2+x
   ans

   d. $$\frac{\dfrac{1}{x}-\dfrac{1}{x+1}}{\dfrac{1}{x}+\dfrac{1}{x+1}}$$
   $\frac{1}{2x+1}$
   ans

   e. $$\frac{\dfrac{2}{x}-\dfrac{1}{x}}{\dfrac{1-y}{y}}$$
   $\frac{y}{x-xy}$
   ans

2. In each case, use the definition of the derivative to find $f'(x)$ (in other words, take the derivative!)

a. $f(x) = 3x - 5$
3
ans
b. $f(x) = \frac{1}{2}x + 1$
$1/2$
ans
c. $f(x) = 2x^2$
$4x$
ans
d. $f(x) = (x^2 + x)$
$2x + 1$
ans
e. $\frac{d}{dx}(e^x)$ (hint: from yesterday's homework, we have $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$)
e^x
ans
f. $f(x) = x^3$
x^3
ans

3. In each case, use the definition of the derivative to find $f'(x)$ (in other words, take the derivative!). Each of these is like one of the "hard problems" (click here)

a. $f(x) = 2x^4$
$8x^3$
ans
b. $f(x) = \sqrt{2x}$
$\frac{1}{\sqrt{2x}}$
ans
c. $f(x) = \frac{2}{x}$
$-\frac{2}{x^2}$
ans
d. $f(x) = \sqrt{x + 1}$
$\frac{1}{2\sqrt{x+1}}$
ans
e. $f(x) = \frac{1}{x+1}$
$-\frac{1}{(x+1)^2}$
ans
f. $f(x) = \frac{1}{\sqrt{x}}$
$\frac{-1}{2x\sqrt{x}}$
ans

4. Recall the derivative of $f(x) = x^2$ is given by $2x$.

a. Show that the derivative of $g(x) = x^2 + 1$ is $2x$ using the definition of the

derivative. Can you find an intuitive reason why $f(x)$ and $g(x)$ would have the same derivative?
Adding a constant moves the curve up or down, but that shift does not affect the slope of the tangent line
ans

b. Find another function whose derivative is $2x$, other than $f(x)$ and $g(x)$.
$x^2 + c$ for any value c
ans

Chapter 18

PROJECT: KILLDEER MIGRATION SPEED

Purpose: To calculate a derivative with messy and incomplete data.

(image credit: Becky Matsubara)

Killdeer are small birds famous for their loud, squeaky call and defense behavior that includes feigning wing injuries. Their population is also mysteriously declining, so getting a handle on their numbers and migratory patterns is important. They are partially migratory, with some going from the southern US and Mexico up to the northern US and Canada, while some stay put. The main question here is to use messy data to get some sort of idea of how fast they are moving at any given time.

1. To simplify things, suppose there are just three migrating killdeer birds, and they don't migrate together necessarily. These three birds were seen at various positions along a

migration route. **Note that each data point is only one of the three birds, and you don't know which one.**

day	0	10	10	15	20	20	25	35
miles	0	10	45	165	75	200	105	200

day	40	50	50	65	70	80	80
miles	310	220	380	405	305	425	445

The goal is to fill out a new table with the velocity or derivative of the killdeer at any given time. Since you don't really know which killdeer is which, this velocity will represent some sort of average or aggregate velocity for the three killdeer.

approximately day	20	40	60
$\frac{\text{miles}}{\text{day}}$			

What are some different ways you can do this? Talk about it as a group, and then work towards filling in the second table. Do the birds seem to be getting faster as time passes, or slower?

2. Same type of question with a new setup. The goal is to figure out a rough speed of migration (just one number this time). There are three areas: Mexico, Oklahoma, and Montana, and you have rough esimates for the abundance in each of the three areas during both the non-breeding and breeding seasons. For simplicity, assume all the killdeer stay within these three areas (even though that is clearly false). How can you figure some sort of average velocity of the killdeer in this case?

Mexico, 1500 miles north of the equator	Abundance
Day 0 (non-breeding)	970
Day 150 (breeding)	260

Oklahoma, 2500 miles north of the equator	Abundance
Day 0 (non-breeding)	150
Day 150 (breeding)	420

Montana, 3200 miles north of the equator	Abundance
Day 0 (non-breeding)	0
Day 150 (breeding)	540

3. Look at the excel spreadsheet entitled "Killdeer abundance data", with data taken from the eBird data repository.

   Click here for a cool visualization of this data.

   Use the data in the spreadsheet to create a best guess average velocity during the time period starting with the non-breeding season and ending with the breeding season (roughly January through June). The final answer is just that one number, but there is a lot of data to sort through to get there!

Data source: Fink, D., T. Auer, A. Johnston, M. Strimas-Mackey, O. Robinson, S. Ligocki, B. Petersen, C. Wood, I. Davies, B. Sullivan, M. Iliff, S. Kelling. 2020. eBird Status and Trends, Data Version: 2018; Released: 2020. Cornell Lab of Ornithology, Ithaca, New York. https://doi.org/10.2173/ebirdst.2018

PART III

RULES FOR DERIVATIVES

CHAPTER 19

ALGEBRA TIPS AND TRICKS PART V (EXPONENTS)

EXPONENTS

When simplifying exponents, remember the exponentiation is just repeated multiplications. So if you have something like

$$x^3x^7$$

This is three xs multiplied by seven xs, so that's ten xs all multiplied together.

$$x^3x^7 = x^{10}$$

Similarly, all these other rules don't even have to be memorized if you just think about how repeated multiplication would work. But here they are anyway.

$$\begin{aligned} A^xA^y &= A^{x+y} \\ \frac{A^x}{A^y} &= A^{x-y} \\ (A^x)^y &= A^{xy} \\ A^{-x} &= \frac{1}{A^x} \\ A^0 &= 1 \end{aligned}$$

Some examples:

- Problem $\frac{e^{11}}{e^5e^4}$.

Here we have eleven es, and we are taking away via division five of them then four of them. Hence we have two es left over: $\frac{e^{11}}{e^5e^4} = \boxed{e^2}$. Note that e is a fundamental constant in mathematics, like π, equal to $2.718281828459045\ldots$ approximately, but we just use e for the exact value.

- Problem $\frac{(A^4B)^3}{(AB^4)^2}$.

We see $(A^4B)^3 = A^{12}B^3$. On bottom, we have $(AB^4)^2 = A^2B^8$. This give $\frac{A^{12}B^3}{A^2B^8}$. Once we get cancel two of the As and three of the Bs, we have $\boxed{\dfrac{A^{10}}{B^5}}$.

- Problem $\left(\frac{\sqrt{2}+\sqrt{3}+\sqrt{5}+\sqrt{7}}{\sqrt[4]{104942}}\right)^{x-x}$.

This looks ugly, but $x - x = 0$, and anything to the zeroth power is 1. Hence the answer is $\boxed{1}$.

- Problem $a^{-5}a^2$.

This combines as $a^{-5+2} = a^{-3}$, which we can also write as $\boxed{\dfrac{1}{a^3}}$.

FRACTIONAL EXPONENTS

One more rule before you go: $A^{n/m} = \left(\sqrt[m]{A}\right)^n$. In other words, a fraction in the exponent is the same thing as taking a square root, cube root, 4th root, etc, depending on what the denominator is. Some examples:

- Problem $25^{1/2}$.

We see this is the same thing as $\sqrt{25}$, which is $\boxed{5}$.

- Problem $8^{2/3}$.

This is the same thing as $\left(\sqrt[3]{8}\right)^2$. We see that $\sqrt[3]{8} = 2$, and hence we do $2^2 = 4$. So $8^{2/3} = \boxed{4}$
.

Chapter 20

POWER RULE

"I've got the power!" –Snap (German band)

If you apply the definition of the derivative to several functions, you'll see:

$$\begin{array}{ll} \frac{d}{dx}x^0 & 0 \\ \frac{d}{dx}x^1 & 1 \\ \frac{d}{dx}x^2 & 2x \\ \frac{d}{dx}x^3 & 3x^2 \\ \frac{d}{dx}x^4 & 4x^3 \end{array}$$

See the pattern?

$$\boxed{\frac{d}{dx}x^n = nx^{n-1}}$$

As we'll see in the next section, this even works for non-integers. **The key is to multiply by the exponent, then decrease the exponent by one.**

The next rules say that constant multiples and addition work nicely.

$$\boxed{\frac{d}{dx}c \cdot f(x) = cf'(x)}$$

$$\boxed{\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)}$$

The key is you can just worry about the derivative of each piece of a sum separately. Constant multiples "come along for the ride". With only these three rules, you can now take the derivative of *any* polynomial. Check it out.

Example **Power**

Find the following derivatives.

1. Problem $\frac{d}{dx}3x^2$

To do this one, we use the power rule on the x^2 part, and get $2x$. However, we are also multiplying by 3, so the answer is multiplied by 3 as well. Hence the answer is $3(2x) = \boxed{6x}$.

2. Problem $\frac{d}{dx}x^3 + x$

By the power rule, we find $\frac{d}{dx}x^3 = 3x^2$, and $\frac{d}{dx}x$ is $\frac{d}{dx}x^1$ which becomes $1x^0$ by the power rule, which is 1. By the addition rule, we have $\frac{d}{dx}x^3 + x = \boxed{3x^2 + 1}$.

3. Problem $\frac{d}{dx}2x^3 + 5$

You take the derivative of x^3 and you have $3x^2$. Times by 2, that leaves $6x^2$. Okay, about the five? It is tempting to leave the five put, but actually $\frac{d}{dx}5 = 0$. Why? Well, it's a constant, so it does not affect the slope. Hence we get $\boxed{6x^2}$.

MORE POWER RULE EXAMPLES

Note that the power rule works with fractional and negative exponents as well! Here are some examples.

Example **Fractional**

Problem Find $\frac{d}{dx}x^{3/2}$.

To apply the power rule in this case, we need to first multiply by the exponent ($3/2$), then subtract one from the exponent $3/2 - 1 = 1/2$. Then we have

$$\frac{d}{dx}x^{3/2} = \frac{3}{2}x^{1/2} = \boxed{\frac{3}{2}x^{1/2}}$$

Example **Negative**

Problem Find $\frac{d}{dx}7x^{-2}$.

In this problem, we just worry about the x^{-2} to start with. We multiply by the exponent (-2), and then subtract one from that to get $-2 - 1 = -3$. Then we have

$$\frac{d}{dx}7x^{-2} = 7(-2x^{-3}) = \boxed{-14x^{-3}}$$

Sometimes there are *hidden* fractional or negative exponents. Don't let them fool you, they are just like the examples above. Just remember these rules:

$$\boxed{\frac{1}{x^n} = x^{-n}} \quad \boxed{\sqrt[m]{x^n} = \sqrt[m]{x}^n = x^{n/m}}$$

Let's see some examples.

Example **Roots**

Find

1. Problem $\frac{d}{dx}\sqrt{x}$

   This problem is much easier if we can rewrite the $\sqrt{x}$. This is the same thing as $x^{1/2}$, and hence we have
   $$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \tfrac{1}{2}x^{1/2-1} = \boxed{\frac{1}{2}x^{-1/2}}.$$

2. Problem $\frac{d}{dx}5\sqrt[3]{x}$

   To solve this, it really helps to rewrite $\sqrt[3]{x}$ as $x^{1/3}$. Once you do that, this prolem is just the power rule and constant multiple rule:
   $$\frac{d}{dx}5\sqrt[3]{x} = 5x^{1/3} = 5\left(\tfrac{1}{3}\right)x^{1/3-1} = \boxed{\frac{5}{3}x^{-2/3}}.$$

3. Problem $\frac{d}{dx}\sqrt[5]{x^3} + x^2 + 7$

   Focus on the easy parts first: we know $\frac{d}{dx}x^2 = 2x$, and we know that $\frac{d}{dx}7 = 0$. So we just need to figure out the $\frac{d}{dx}\sqrt[5]{x^3}$. What is this? Well, we can rewrite this as $\frac{d}{dx}x^{3/5}$. So now it is just the power rule, and we multiply by $3/5$ and subtract to get $-2/5$. Hence $\frac{d}{dx}x^{3/5} = \frac{3}{5}x^{-2/5}$. Putting it all together:
   $$\frac{d}{dx}\sqrt[5]{x^3} + x^2 + 7 = \boxed{\frac{3}{5}x^{-2/5} + 2x}.$$

Example **Powers of x in the denominator**

1. Problem $\frac{d}{dx}\frac{1}{x}$

   We can rewrite $\frac{d}{dx}\frac{1}{x} = \frac{d}{dx}\frac{1}{x^1}$ as $\frac{d}{dx}x^{-1}$. Now we apply the power rule:
   $$\frac{d}{dx}\frac{1}{x} = \frac{d}{dx}x^{-1} = \boxed{-1x^{-2}}.$$

2. Problem $\frac{d}{dx}\frac{4}{x^2}$

We can think of 4 as just a constant out front, and hence we want $\frac{d}{dx}4\left(\frac{1}{x^2}\right)$. We can then rewrite $\frac{1}{x^2}$ as x^{-2}. And we want $\frac{d}{dx}4(x^{-2})$. Using the power rule, we multiply by -2 and subtract one, and we have

$$\frac{d}{dx}\frac{4}{x^2} = \frac{d}{dx}4x^{-2} = \boxed{-8x^{-3}}.$$

3. Problem $\frac{d}{dx}\frac{1}{\sqrt{x}}$

This combines the fractional and denominator stuff. We first rewrite $\sqrt{x}$ as $x^{1/2}$:

$$\frac{d}{dx}\frac{1}{\sqrt{x}} = \frac{d}{dx}\frac{1}{x^{1/2}}.$$

We then rewrite as a negative fractional exponent.

$$\frac{d}{dx}\frac{1}{\sqrt{x}} = \frac{d}{dx}x^{-1/2}$$

Finally, we use the power rule.

$$\frac{d}{dx}\frac{1}{\sqrt{x}} = \boxed{-\frac{1}{2}x^{-3/2}}.$$

Chapter 21

HOMEWORK: POWER RULE

1. Compute the following derivatives. Do not use the definition of the derivative. Instead, use the linearity and power rules we talked about in this section.

   a. $\dfrac{d}{dx}\, x^{15}$
   $15x^{14}$
   ans

   b. $\dfrac{d}{dx}\, 3x^{6}$
   $18x^{5}$
   ans

   c. $\dfrac{d}{dx}\, \dfrac{1}{2}x^{4}$
   $2x^{3}$
   ans

   d. $\dfrac{d}{dx}\, 3x^{2} + 6x - 1$
   $6x + 6$
   ans

   e. $\dfrac{d}{dx}\, (2x + 3)^{2}$
   $8x + 12$
   ans

   f. $\dfrac{d}{dx}\, x^{1/3}$
   $\frac{1}{3}x^{-2/3}$
   ans

   g. $\dfrac{d}{dx}\, 7x^{-4}$
   $-28x^{-5}$
   ans

   h. $\dfrac{d}{dx}\, 2x^{-1/2} + 4x^{1/2}$
   $-x^{-3/2} + 2x^{-1/2}$
   ans

i. $\dfrac{d}{dx}\sqrt{x}$
$\frac{1}{2}x^{-1/2}$
ans

j. $\dfrac{d}{dx}\frac{1}{x}$
$-x^{-2}$
ans

k. $\dfrac{d}{dx}\sqrt[3]{x^2}$
$\frac{2}{3}x^{-1/3}$
ans

l. $\dfrac{d}{dx}\frac{2}{\sqrt{x}}$
$-x^{-3/2}$
ans

CHAPTER 22

ALGEBRA TIPS AND TRICKS PART VI (LOGARITHMS)

LOGARITHMS

A logarithm is the inverse function to an exponential function. For example, for the exponential function $y = 2^x$, if we have an input of $x = 6$, we get an output of $y = 64$, and we write $64 = 2^6$. The logarithmic function $y = \log_2(x)$ is the reverse of this. We swap the input and the output, so now $x = 64$ and $y = 6$. We see $6 = \log_2(64)$.

In calculus, we will mostly use the exponential function e^x and its inverse, $\ln(x)$. Below are some important formulas:

$$\begin{aligned} e^{\ln(x)} &= x \\ \ln(e^x) &= x \\ \ln(x) + \ln(y) &= \ln(xy) \\ \ln(x) - \ln(y) &= \ln\left(\frac{x}{y}\right) \\ a\ln(x) &= \ln(x^a) \end{aligned}$$

Examples:

Problem $\ln(x^2) - \ln(x)$.

There are two ways to do this one. First, we can bring down the exponent of two down in front $\ln(x^2) = 2\ln(x)$. Then can combine the like terms of $2\ln(x)$ and $\ln(x)$:

$$\begin{aligned} \ln(x^2) - \ln(x) &= 2\ln(x) - \ln(x) \\ &= \boxed{\ln(x)} \end{aligned}$$

Alternatively, we can rewrite the subtraction as a division, like so:

$$\begin{aligned}\ln(x^2) - \ln(x) &= \ln\left(\frac{x^2}{x}\right) \\ &= \boxed{\ln(x)}\end{aligned}$$

Either way we get the same answer!

Problem $\ln(e^3x^4) - 3\ln(x)$.

First, we rewrite the multiplication using addition. Then we can simply from there.

$$\begin{aligned}\ln(e^3x^4) - 3\ln(x) &= \ln(e^3) + \ln(x^4) - 3\ln(x) \\ &= 3 + 4\ln(x) - 3\ln(x) \\ &= \boxed{3 + \ln(x)}\end{aligned}$$

Problem $\ln(\sqrt{x})$.

We know $\sqrt{x} = x^{1/2}$, so $\ln\left(\sqrt{x}\right) = \ln\left(x^{1/2}\right) = \boxed{\frac{1}{2}\ln(x)}$.

Problem $\ln\left(\frac{\sqrt{x}y}{z^3}\right) - \ln\left(\frac{z}{\sqrt{x}y^3}\right)$.

We can rewrite all the products and divisions as addition and subtraction:

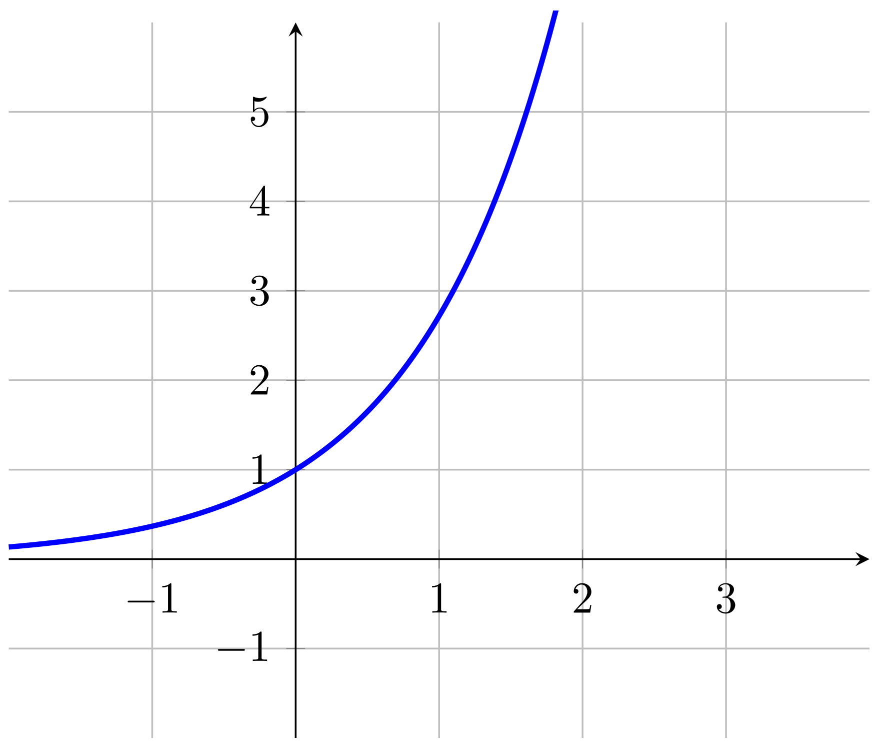
$$\begin{aligned}\ln\left(\frac{\sqrt{x}y}{z^3}\right) - \ln\left(\frac{z}{\sqrt{x}y^3}\right) &= \ln(\sqrt{x}) + \ln(y) - \ln(z^3) - [\ln(z) - \ln(\sqrt{x}) - \ln(y^3)] \\ &= \frac{1}{2}\ln(x) + \ln(y) - 3\ln(z) - \ln(z) + \frac{1}{2}\ln(x) + 3\ln(y) \\ &= \boxed{\ln(x) + 4\ln(y) - 4\ln(z)}.\end{aligned}$$

Chapter 23

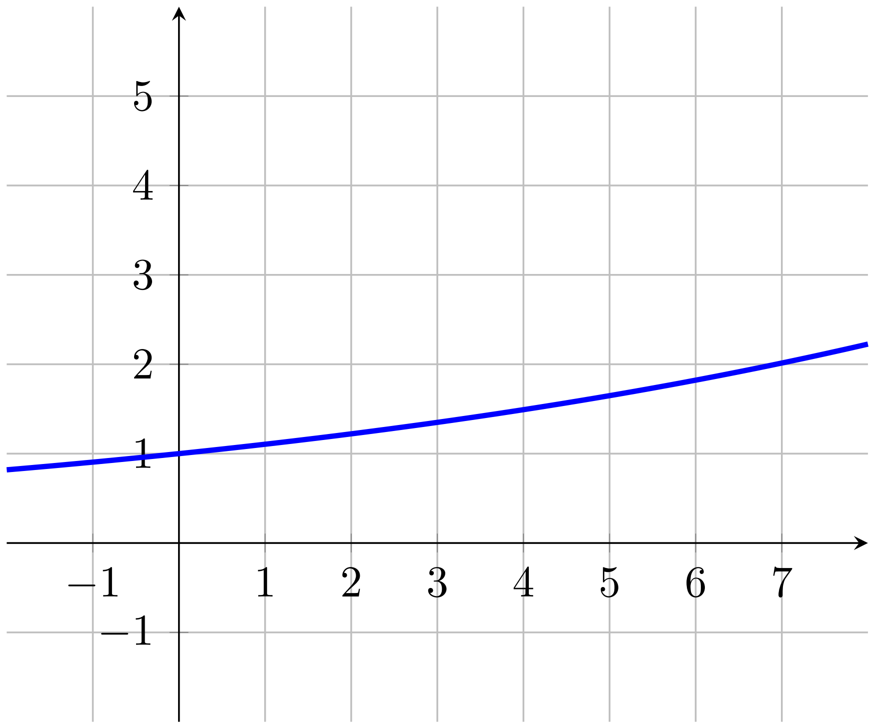
EXPONENTIALS, LOGARITHMS, AND TRIG FUNCTIONS

THE FUNCTION e^x

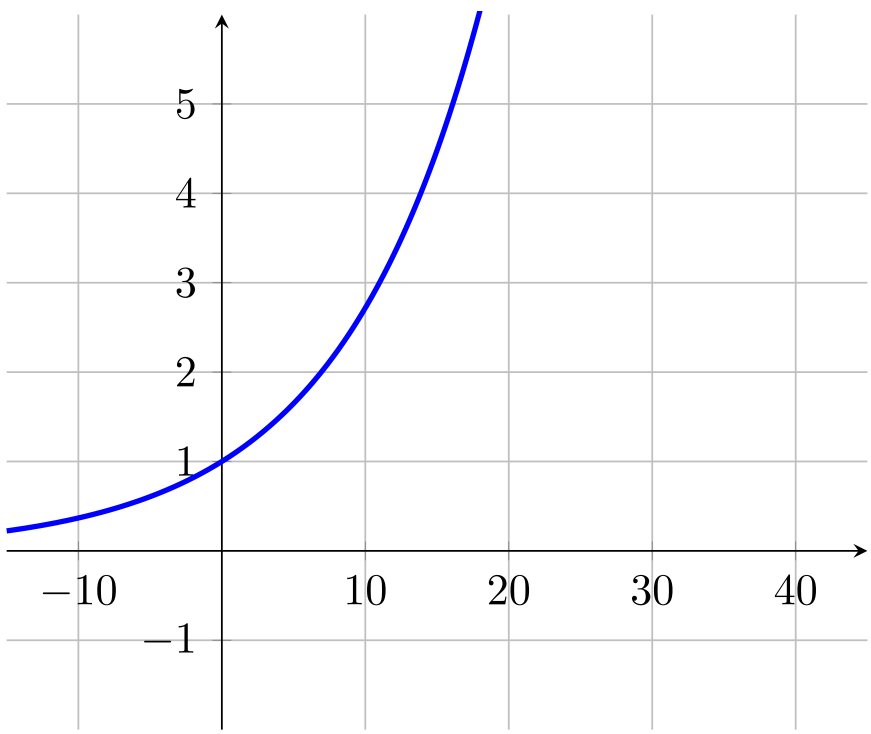
Recall that exponential functions like 2^x, 3^x, and e^x. Note that e is just a number, equal to about 2.718, and is very special when it is the base of an exponential function. All these exponential functions grow extremely quickly. Here is e^x, and watch how quickly it flies out of the picture.



We can modify it so it doesn't grow so fast. Consider $e^{0.1x}$:



But even this starts to grow very quickly when x gets large. Here is $e^{0.1x}$ again for larger values of x.



Hey, that looks a lot like the graph of e^x did! Why is that?

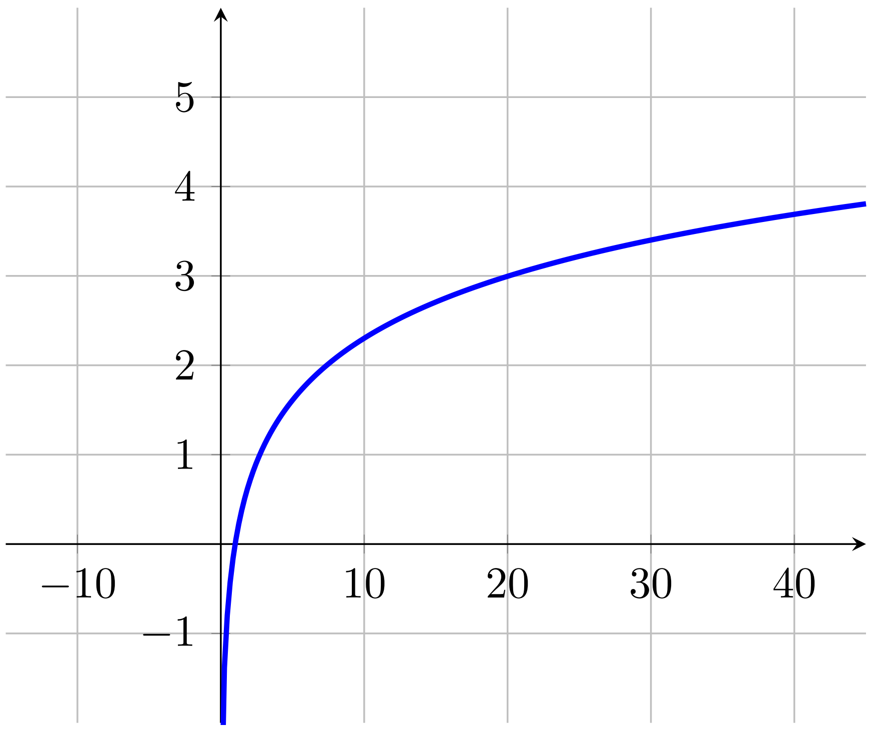
Exponentials at various rates of growth model a wide array of phenomena, including population growth, economic growth, radioactive decay, and more. And the reason e^x is a very special function is one of the most amazing formulas in math:

$$\boxed{\frac{d}{dx}e^x = e^x}$$

That's right; e^x doesn't change when you take the derivative!

THE FUNCTION $\ln(x)$

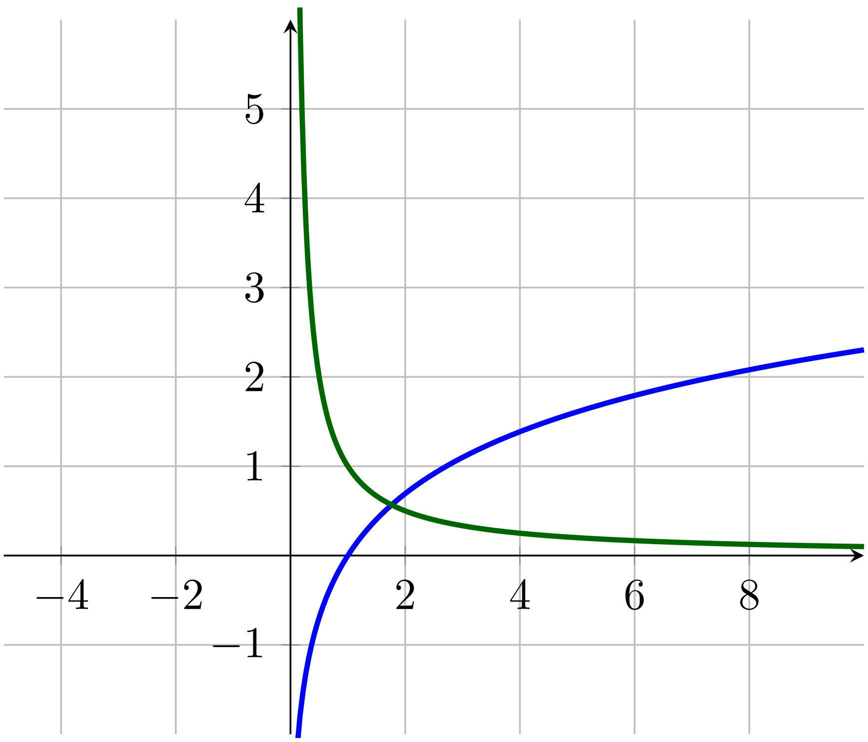
Logarithms, on the other hand, are some of the slowest growing functions. Here is $\ln(x)$, which is $\log_e(x)$, for large values of x:



Notice even for large values of x, the function does not get larger than 4 in this picture. Natural log $\ln(x)$, which again is the log with base e, also has a special derivative.

$$\boxed{\frac{d}{dx}\ln(x) = \frac{1}{x}}$$

Here is $\ln(x)$ in blue plotted with its derivative $\frac{1}{x}$ in green.



Example **Exponents and Logs and More**
Find the following derivatives.

1. Problem $\frac{d}{dx}(e^x + \ln(x))$

We just need to take the derivative of each term. The e^x stays the same when you take the derivative, so we just leave that piece. The $\ln(x)$, as we saw above, has derivative $\frac{1}{x}$. Hence

$$\frac{d}{dx}(e^x + \ln(x)) = \boxed{e^x + \frac{1}{x}}.$$

2. Problem $\frac{d}{dx}(5x^2 + 3e^x)$

We can use the power rule on $5x^2$ — multiply by the two, and subtract one from the two, to get $10x$. We then see that e^x is e^x, and the 3 stays along for the ride. So

$$\frac{d}{dx}(5x^2 + 3e^x) = \boxed{10x + 3e^x}.$$

3. Problem $\frac{d}{dx}\left(\frac{1}{x} + 4\ln(x)\right)$

Remember that to take the derivative of $\frac{1}{x}$, we rewrite as x^{-1} and use the power rule, and we have $-1x^{-2}$. For $4\ln(x)$, the $\ln(x)$ becomes $\frac{1}{x}$, and the four multiplies. Therefore we have

$$\frac{d}{dx}\left(\frac{1}{x}+4\ln(x)\right)=-1x^{-2}+4\left(\frac{1}{x}\right)$$

But wait — these fraction actually can be added together. First, change the x^{-2} back into $\frac{1}{x^2}$. Then we'll get a common denominator, and simplify.

$$\begin{aligned}-1x^{-2}+4\left(\frac{1}{x}\right)&=\frac{-1}{x^2}+\frac{4}{x}\\&=\frac{-1}{x^2}+\frac{4}{x}\cdot\frac{x}{x}\\&=\frac{-1}{x^2}+\frac{4x}{x^2}\\&=\frac{4x-1}{x^2}\end{aligned}$$

Hence we have

$$\frac{d}{dx}\left(\frac{1}{x}+4\ln(x)\right)=\boxed{\frac{4x-1}{x^2}}.$$

Two more quick formulas.

$$\boxed{\frac{d}{dx}a^x=\ln(a)\cdot a^x}$$

$$\boxed{\frac{d}{dx}\log_a(x)=\frac{1}{\ln(a)\cdot x}}$$

We'll prove the first rule once we have the chain rule up and running. For the second rule, the proof only requires the base change formula.

Example **Log in base a proof**

Problem Prove the following rule

- $\dfrac{d}{dx}\log_a(x)=\frac{1}{\ln(a)\cdot x}$

We use the base change formula, which states

$$\log_a(x)=\frac{\ln(x)}{\ln(a)}=\frac{1}{\ln(a)}\ln(x)$$

From this we see

$$\begin{aligned}\frac{d}{dx}\ \log_a(x) &= \frac{d}{dx}\ \frac{1}{\ln(a)}\ln(x)\\ &= \frac{1}{\ln(a)}\frac{d}{dx}\ \ln(x)\\ &= \frac{1}{\ln(a)}\cdot\frac{1}{x}\\ &= \frac{1}{\ln(a)\cdot x}.\end{aligned}$$

Now this proof demonstrates a tricky thing in calculus: sometimes we can just "bring things out" of the derivative like we did with $\frac{1}{\ln(a)}$, and other times we cannot. The reason we could take the $\frac{1}{\ln(a)}$ out is that it is considered a *constant*. The derivative $\frac{d}{dx}$ is only measuring the change as x changes, not as a changes. So $\frac{1}{\ln(a)}$ is a constant, or unchanging value as x changes. Therefore, by the constant multiple rule, we can just take it out of the derivative.

Now to use the new rules.

Example **Base a examples**

Compute the following derivatives.

1. Problem $\frac{d}{dx}\ (2^x + 3^x)$

   We use the formula on 2^x and see the derivative is $\ln(2)\cdot 2^x$. Same goes for $\frac{d}{dx}\ 3^x = \ln(3)\cdot 3^x$. So the end result is $\boxed{\ln(2)\cdot 2^x + \ln(3)\cdot 3^x}$.

2. Problem $\frac{d}{dx}\ 3\log_2(x)$

   We use the formula to get $\frac{d}{dx}\ \log_2(x) = \frac{1}{\ln(2)x}$, and multiply the answer by 3 and get $\boxed{\dfrac{3}{\ln(2)x}}$.

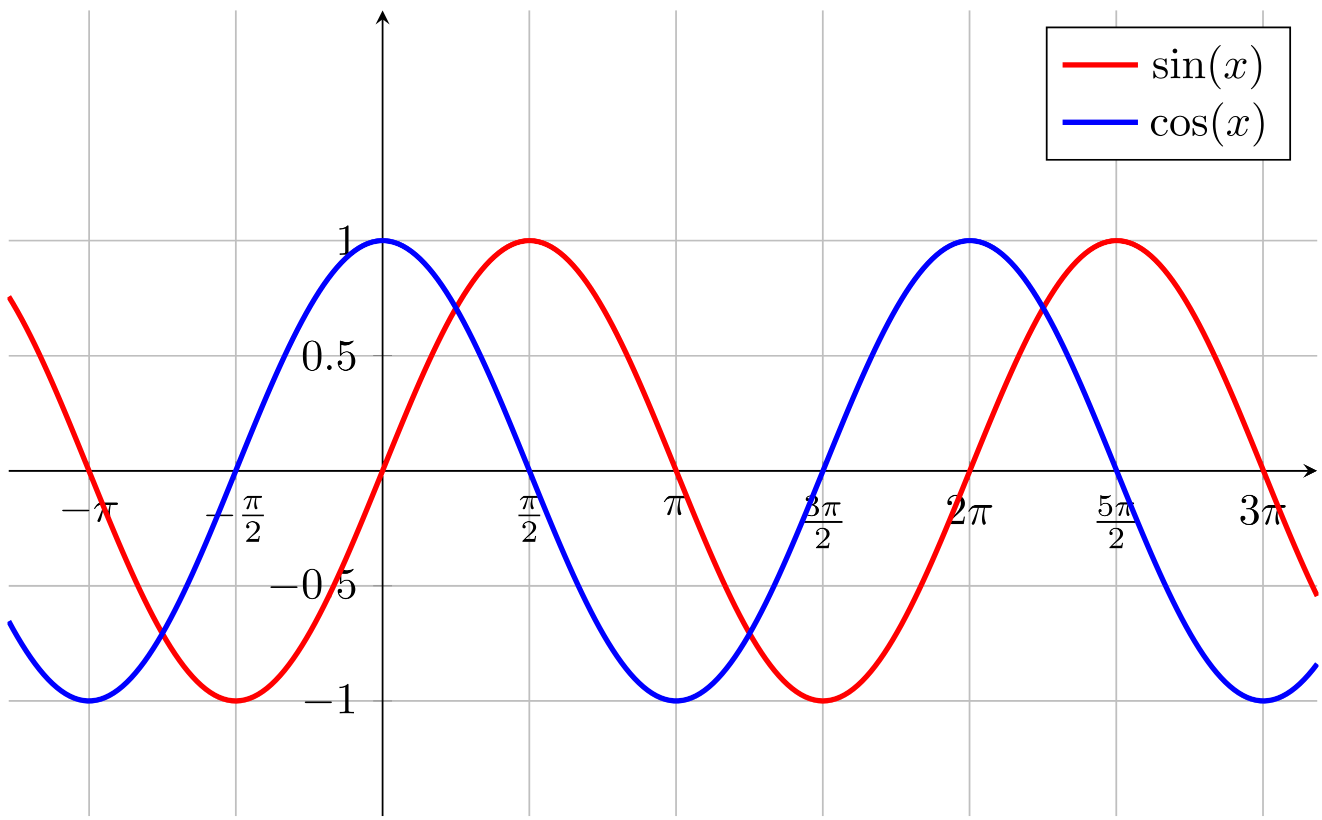
3. Problem $\frac{d}{dx}\ 5^x/\ln(5)$

We have

$$\begin{aligned}\frac{d}{dx}\frac{5^x}{\ln(5)} &= \frac{1}{\ln(5)}\frac{d}{dx}5^x \\ &= \frac{1}{\ln(5)}\ln(5)\cdot 5^x \\ &= \boxed{5^x}\end{aligned}$$

THE FUNCTIONS $\sin(x)$ **AND** $\cos(x)$

The sine function, denoted $\sin(x)$, captures oscillating behavior of waves, circles, pendulums, and more. Cosine, denoted $\cos(x)$, is a similar function that does the same thing. Here are $\sin(x)$ and $\cos(x)$ on a graph.



A few important values to know for $\sin(x)$ and $\cos(x)$ are given by the table below.

$\mathbf{x}$	$\sin(\mathbf{x})$	$\mathbf{cos}(\mathbf{x})$
0	0	1
$\pi/2$	1	0
π	0	-1
$3\pi/2$	-1	0
2π	0	1

First note that the x values have a π in them — this is actually a way of measuring angles called radians. You can also use sin and cos with degree angle measurements, but for calculus, it works much better in radians. **Make sure your calculator is in radian mode for this class.** Notice also that the outputs for $\sin(x)$ and $\cos(x)$ with $x = 0$ are the same as with $x = 2\pi$. That's not a coincidence — sin and cos functions repeat over and over again every $\Delta x = 2\pi$.

The functions $\sin(x)$ and $\cos(x)$ work very well with calculus, as shown by these important formulas:

$$\boxed{\frac{d}{dx}\sin(x) = \cos(x)}$$

$$\boxed{\frac{d}{dx}\cos(x) = -\sin(x)}$$

Example **Sine and Cosine derivatives**

Compute the following:

1. Problem $\frac{d}{dx} \quad -2\cos(x)$

   Using the formulas from this section and earlier in the chapter, we see
   $$\frac{d}{dx} \quad -2\cos(x) = -2(-\sin(x)) = \boxed{2\sin(x)}$$

2. Problem $\frac{d}{dx}(\sin(x) + \cos(x) + e^x + \ln(x))$

   $$\frac{d}{dx}(\sin(x) + \cos(x) + e^x + \ln(x)) = \boxed{\cos(x) - \sin(x) + e^x + \frac{1}{x}}.$$

Chapter 24

HOMEWORK: EXPONENTS, LOGS, TRIG FUNCTIONS

1. Take the derivative of the following functions

   a. $\dfrac{d}{dx}\, x^3 + e^x$
   $3x^2 + e^x$
   ans

   b. $\dfrac{d}{dx}\, 5\ln(x) - x^{-1}$
   $\frac{5}{x} + x^{-2}$
   ans

   c. $\frac{d}{dx} \log_{10}(x)$
   $\frac{1}{\ln(10)x}$
   ans

   d. $\frac{d}{dx}\, 5\sin(x) - 3\cos(x)$
   $5\cos(x) + 3\sin(x)$
   ans

Chapter 25

PRODUCT RULE

Okay, but what about $\frac{d}{dx}x \cdot e^x$? Can we just take the derivative of each like this?

$$\begin{aligned}\frac{d}{dx}x \cdot e^x &= \left(\frac{d}{dx}x\right) \cdot \left(\frac{d}{dx}e^x\right) \\ &= 1 \cdot e^x = e^x\end{aligned}$$

Unfortunately, no. **Just to be clear: the above calculation is false!**

Think about $\frac{d}{dx}x \cdot x$. We know $\frac{d}{dx}x = 1$, so therefore you might think

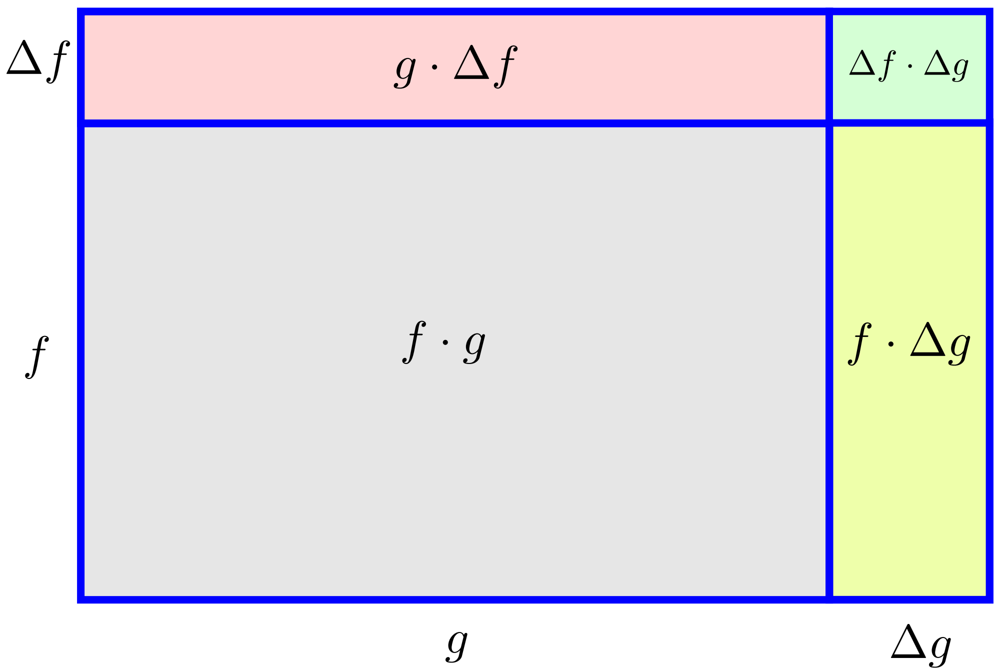
$$\frac{d}{dx}x \cdot x = 1 \cdot 1 = 1$$

But we also know that $\frac{d}{dx}x \cdot x = \frac{d}{dx}x^2 = 2x$. So what do we make of this? Well, we just have to give up on the idea of "taking the derivative of each" with products. Luckily, there is a rule called the *product rule* that works great:

$$\boxed{\frac{d}{dx}f \cdot g = fg' + gf'}$$

The motto for this rule is "the first times the derivative of the second, plus the second times the derivative of the first."

Where does this rule come from? Well, consider this picture:



Here we have a rectangle. The height is f, the width is g. The area of the rectangle is $f \cdot g$. If we were to make f bigger by a little bit, by say Δf, and g bigger by Δg, then the rectangle would become bigger too. *We want to find out how quickly the area is increasing.* So how quickly is it increasing? You can see in the picture we add three sections on: $f \cdot \Delta g$, $g \cdot \Delta f$, and $\Delta f \cdot \Delta g$. If you think of Δf and Δg as being really small, though, the $\Delta f \cdot \Delta g$ terms is incredibly tiny — so tiny it does not affect the answer in the limit. Therefore, the new area is $f \cdot \Delta g + g \cdot \Delta f$. This is where the product rule comes from — it's how an area changes as you change each side length.

Let's see this rule in action.

Example

Product Rule with xe^x

Problem Find $\frac{d}{dx} xe^x$.

In this case, we can attack this using the product rule with $f = x$, and $g = e^x$. We can easily take the derivative of each part: $f' = 1$, and $g' = e^x$. Hence, using the formula $\frac{d}{dx} fg = fg' + gf'$, we have

$$fg' + gf' = xe^x + e^x(1) = \boxed{xe^x + e^x}.$$

Example

Another Product Rule

Problem Find $\frac{d}{dx} x^2 \cos(x)$.

We see $f = x^2$, $f' = 2x$ and $g = \cos(x)$, $g' = -\sin(x)$. Hence, we have

$$\begin{aligned} \frac{d}{dx} x^2 \cos(x) &= fg' + gf' \\ &= x^2(-\sin(x)) + \cos(x) 2x \\ &= \boxed{-2x^2 \sin(x) + 2x\cos(x)}. \end{aligned}$$

Example

Product Rule with $x \cdot x$

Problem Find $\frac{d}{dx} x \cdot x$ in two different ways: one way using the product rule, one way using the power rule.

Using the power rule, we see $\frac{d}{dx} x \cdot x = \frac{d}{dx} x^2 = 2x$. Using the product rule, we set $f = x$ and $g = x$. In which case, $f' = 1$ and $g' = 1$. Thus

$$fg' + gf' = (x)(1) + (x)(1) = \boxed{2x}.$$

Hurray! The same answer! Isn't it great when math just plain works.

Example

Another Product Rule

Problem Find $\frac{d}{dx} \sqrt{x} \ln(x)$.

We see $f = \sqrt{x}$, $g = \ln(x)$. For f', we rewrite $f = x^{1/2}$ and use the power rule to get $f' = \frac{1}{2} x^{-1/2}$, which is $f' = \frac{1}{2\sqrt{x}}$. As we saw in the exponents and logarithms section, $g' = \frac{1}{x}$. Hence

$$\begin{aligned} \frac{d}{dx} \sqrt{x} \ln(x) &= fg' + gf' \\ &= \sqrt{x} \frac{1}{x} + \ln(x) \frac{1}{2\sqrt{x}} \\ &= \frac{\sqrt{x}}{x} + \frac{\ln(x)}{2\sqrt{x}} \end{aligned}$$

Now if you want to be a bit fancy with the algebra, we can simplify $\frac{\sqrt{x}}{x}$ using rational exponents. It is equal to $\frac{x^{1/2}}{x^1}$, which is $x^{1/2-1} = x^{-1/2}$. If we turn this back into a root, we get $\frac{1}{\sqrt{x}}$. From here, we can find a common denominator and add the two fractions.

$$\begin{aligned}
\frac{d}{dx}\sqrt{x}\ln(x) &= \frac{1}{\sqrt{x}} + \frac{\ln(x)}{2\sqrt{x}} \\
&= \frac{2}{2\sqrt{x}} + \frac{\ln(x)}{2\sqrt{x}} \\
&= \boxed{\frac{\ln(x)+2}{2\sqrt{x}}}.
\end{aligned}$$

There you go.

Chapter 26

HOMEWORK: PRODUCT RULE

1. Take the derivatives of the following functions.

   a. $f(x) = x^2 e^x$
   $f'(x) = x^2 e^x + 2xe^x$
   ans

   b. $g(x) = (\sqrt{x} + 1)\left(x^2\right)$
   $g'(x) = \frac{5}{2}x^{3/2} + 2x$
   ans

   c. $h(x) = \frac{1}{x}\left(e^x + 1\right)$
   $h'(x) = -\frac{1}{x^2}(e^x + 1) + \frac{1}{x}e^x$
   ans

   d. $i(x) = \ln(x)x$
   $i'(x) = 1 + \ln(x)$
   ans

   e. $j(x) = \sqrt{x}\ln(x)$
   $j(x) = \frac{1}{2\sqrt{x}}\ln(x) + \frac{1}{\sqrt{x}}$
   ans

   f. $m(x) = (x^3 + x^2)\sin(x)$
   $m'(x) = (3x^2 + 2x)\sin(x) + (x^3 + x^2)\cos(x)$
   ans

   g. $l(x) = e^{2x}$ (Hint: You can rewrite this as $e^x \cdot e^x$)
   $l'(x) = 2e^{2x}$
   ans

   h. $\ell(x) = xe^x \ln(x)$
   $\ell'(x) = e^x \ln(x) + e^x \ln(x) + e^x$
   ans

CHAPTER 27

QUOTIENT RULE

What about $\frac{d}{dx}\left(\frac{x}{e^x}\right)$? Can we just take the derivative of the top and bottom separately, and put them together? Nope, we need a *quotient rule*.

$$\boxed{\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{gf' - fg'}{g^2}}$$

Where did this strange formula come from? Some fancy algebra will get you there as the next example shows.

Example **Proving the Quotient Rule**

Problem Prove the quotient rule $\frac{d}{dx}\left(\frac{a}{b}\right) = \frac{ba'-ab'}{b^2}$.

Note that I'm using a and b instead of f and g right now because we will need f and g to mean something else in just a second.

So what can we do? One way it to use the product rule in a strange manner. We are going to apply it to $\frac{d}{dx}\left(b \cdot \frac{a}{b}\right)$. We set $f = b$, and $g = \frac{a}{b}$. We see

$$\begin{aligned}\frac{d}{dx}\left(b \cdot \frac{a}{b}\right) &= fg' + g'f \\ &= b\left(\frac{d}{dx}\frac{a}{b}\right) + \left(\frac{a}{b}\right)\left(\frac{d}{dx}b\right) \\ &= b\left(\frac{d}{dx}\frac{a}{b}\right) + \frac{ab'}{b}\end{aligned}$$

But notice that $\frac{d}{dx}\left(b \cdot \frac{a}{b}\right) = \frac{d}{dx}a = a'$. Hence, we have

$$a' = b\left(\frac{d}{dx}\frac{a}{b}\right) + \frac{ab'}{b}$$

Now we just have to solve for $\frac{d}{dx}\frac{a}{b}$, and we have a formula for derivatives of quotients!

$$
\begin{aligned}
a' &= b\left(\frac{d}{dx}\frac{a}{b}\right) + \frac{ab'}{b} \\
a' - \frac{ab'}{b} &= b\left(\frac{d}{dx}\frac{a}{b}\right) \\
\frac{1}{b}\left(a' - \frac{ab'}{b}\right) &= \frac{d}{dx}\frac{a}{b} \\
\frac{a'}{b} - \frac{ab'}{b^2} &= \frac{d}{dx}\frac{a}{b} \\
\frac{ba'}{b^2} - \frac{ab'}{b^2} &= \frac{d}{dx}\frac{a}{b} \\
\frac{ba' - ab'}{b^2} &= \frac{d}{dx}\frac{a}{b}
\end{aligned}
$$

If we turn this equation around, it gives the same quotient rule I mentioned earlier:

$$\boxed{\frac{d}{dx}\left(\frac{a}{b}\right) = \frac{ba' - ab'}{b^2}}$$

This has a cute rhyme to it: "low dee high minus high dee low, over the square of what's below". The "low dee high" means ba', since b is the "low" and a' is the "dee high". Then "minus high dee low" is $-ab'$. Finally, "over the square of what's below" is b^2.

Let's see how it looks applying the quotient rule.

Example **Quotient Rule with $\frac{x}{e^x}$**

Problem Find $\frac{d}{dx}\left(\frac{x}{e^x}\right)$.

We set $a = x$ and $b = e^x$. We see $a' = 1$, $b' = e^x$. Using the formula $\dfrac{d}{dx}\left(\dfrac{a}{b}\right) = \frac{ba'-ab'}{b^2}$, we have

$$\begin{aligned}\frac{d}{dx}\left(\frac{x}{e^x}\right) &= \frac{ba' - ab'}{b^2} \\ &= \frac{(e^x)(1) - (x)(e^x)}{(e^x)^2} \\ &= \frac{e^x(1-x)}{(e^x)^2} \\ &= \boxed{\frac{1-x}{e^x}}\end{aligned}$$

Example **Quotient Rule with $\frac{x^2}{x}$**

Problem Find $\frac{d}{dx}\left(\frac{x^2}{x}\right)$ using the quotient rule and power rule.

If we simplify and turn $\frac{x^2}{x}$ into just x, then we have $\frac{d}{dx}\left(\frac{x^2}{x}\right) = \frac{d}{dx}x = \boxed{1}$. Easy enough.

Using the quotient rule, we set $a = x^2$ and $b = x$, with $a' = 2x$ and $b = 1$. We have

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^2}{x}\right) &= \frac{ba' - ab'}{b^2} \\ &= \frac{(x)(2x) - x^2(1)}{(x)^2} \\ &= \frac{2x^2 - x^2}{x^2} \\ &= \frac{x^2}{x^2} \\ &= \boxed{1}\end{aligned}$$

Same thing we got before!

Example **More Quotient Rule**

Problem Find $\frac{d}{dx}\frac{(x^2+2x)}{\ln(x)}$.

In this case, $a = x^2 + 2x$ and $b = \ln(x)$. We have $a' = 2x + 2$, $b' = \frac{1}{x}$. Hence we have

$$\begin{aligned} \frac{d}{dx}\frac{(x^2+2x)}{\ln(x)} &= \frac{ba' - ab'}{b^2} \\ &= \frac{(\ln(x))(2x+2) - (x^2+2x)\left(\frac{1}{x}\right)}{(\ln x)^2} \\ &= \boxed{\frac{\ln(x)(2x+2) - (x+2)}{(\ln x)^2}} \end{aligned}$$

This doesn't really simplify farther, so that's our answer.

Example **With a sine this time**

Problem Find $\frac{d}{dx}\frac{\sin(x)}{x}$.

In this case $a = \sin(x)$ and $b = x$, so $a' = \cos(x)$ and $b' = 1$. Hence

$$\begin{aligned} \frac{d}{dx}\frac{\sin(x)}{x} &= \frac{ba' - ab'}{b^2} \\ &= \boxed{\frac{x\cos(x) - \sin(x)}{x^2}}. \end{aligned}$$

Chapter 28

HOMEWORK: QUOTIENT RULE

1. Find the derivatives of the following functions.

   a. $f(x) = \frac{e^x}{x}$
      $f'(x) = \frac{xe^x - e^x}{x^2}$
      ans

   b. $g(x) = \frac{\sqrt[3]{x}}{\ln(x)}$
      $g'(x) = \frac{\frac{1}{3}x^{-2/3}\ln(x) - \sqrt[3]{x}/x}{(\ln x)^2} = \frac{3x^{-2/3}(\ln(x)-1)}{3(\ln x)^2}$
      ans

   c. $h(x) = \frac{1}{x^2+5x+6}$
      $h'(x) = \frac{-(2x+5)}{(x^2+5x+6)^2}$
      ans

   d. $i(x) = \frac{\cos(x)}{1+x^2}$
      $i'(x) = \frac{-(1+x^2)\sin(x) - 2x\cos(x)}{(1+x^2)^2}$
      ans

   e. $j(x) = \frac{\ln(x)}{x^2}$
      $j'(x) = \frac{1-2\ln(x)}{x^3}$
      ans

   f. $k(x) = e^{-x}$ (How can you write this as a fraction?)
      $k'(x) = \frac{-e^x}{(e^x)^2} = -\frac{1}{e^x}$
      ans

   g. $\ell(x) = \frac{xe^x}{1+x}$
      $\ell'(x) = \frac{x^2e^x + xe^x + e^x}{(1+x)^2}$
      ans

Chapter 29

Chain Rule

With one additional rule, we will have the power to take the derivative of any function we can write down. What is this amazing rule? Why, it's called the *chain rule*. The chain rule is $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$. If we drop the (x) on each function (note that it is still there, it is just implied), we have a slightly shorter version:

$$\boxed{\frac{d}{dx}f(g) = f'(g) \cdot g'}$$

Here, $f(g)$ is called function composition. It does not mean f times g. It means we are sticking g inside of f! Like f is eating g! That's actually cannibalism if you think about it — so don't think about it too closely. But do remember how to do function composition.

Why does this work? I'm not going to do a formal proof, but let's run through an idea behind it. Recall that the derivative is the slope or how steep the graph of a function is. This is a lot easier to think about if we're talking about lines. For example, suppose $f = 6x - 10$, and $g = \frac{1}{2}x + 3$. As we know from algebra, the slope of the f line is 6, and the slope of the g line is $\frac{1}{2}$.

So what is the slope of $f(g)$? What this means is we are putting g inside of f. So if $f = 6x - 10$, and $g = \frac{1}{2}x + 3$, we take the blue $\frac{1}{2}x + 3$ and use that to replace the red x. Here is what is would look like:

$$\begin{aligned}
f(g) &= 6\left(\frac{1}{2}x + 3\right) - 10 \\
&= 6\left(\frac{1}{2}x + 3\right) - 10 \\
&= 3x + 18 - 10 \\
&= 3x - 8
\end{aligned}$$

So again, what is the slope of $f(g)$? We can see from this calculation that it is 3, the product of the two slopes 6 and $\frac{1}{2}$. That's why you have $f' \cdot g'$ in the formula.

Okay, but what about the (g) after the f' in the formula? One way to think about the function composition $f(g)$ is we are looking at the f curve at the x-location of g. When you take the derivative, you're still looking at that same location (now on the f' curve), so you still need that (g) there to specify that location.

Example **Chain rule with** $\ln(x^2+x)$

Problem Find $\frac{d}{dx}\ln(x^2+x)$.

We must identify an "inside" (g) and "outside" (f) function in order to use the chain rule. Often, the "inside" function will be in parentheses (though not always). That works in this case, so the "inside" function is x^2+x, so $g=x^2+x$. The outside function is $\ln$, and hence $f=\ln(x)$. We also know $f'=\frac{1}{x}$ and $g'=2x+1$. Now to use the chain rule, we first need $f'(g)$. What is this? Well, remember that this is not multiplication, but it is sticking one function inside another. In this case, we are taking $g=x^2+x$ and sticking it into the function $f'=\frac{1}{x}$. This means we replace the x in $\frac{1}{x}$, replacing it with x^2+x, and get $f'(g)=\frac{1}{x^2+x}$. Hence we have

$$\begin{aligned}\frac{d}{dx}\ln(x^2+x) &= f'(g)\cdot g'\\ &= \frac{1}{(x^2+x)}\cdot(2x+1)\\ &= \boxed{\frac{2x+1}{x^2+x}}.\end{aligned}$$

There you have it!

Example **Chain rule with** $(3x+1)^2$

Problem Find $\frac{d}{dx}(3x+1)^2$ in two different ways: using power rule, and using the chain rule.

Using the power rule, we first multiply $(3x+1)^2=(3x+1)(3x+1)=9x^2+3x+3x+1=9x^2+6x+1$. In this form, it is easy to find the derivative: $\frac{d}{dx}9x^2+6x+1=18x+6$.

Using the chain rule, we identify the inside g function as $3x+1$, and the outside function as x^2. We then have

$$\begin{aligned}\frac{d}{dx}(3x+1)^2 &= f'(g)\cdot g'\\ &= 2(3x+1)(3)\\ &= 6(3x+1)\\ &= \boxed{18x+6}.\end{aligned}$$

Again, math just works!

Things can get quite complicated with the chain rule.

Example **Complicated chain rule**

Problem Find $\frac{d}{dx}\sqrt[4]{x^2+2e^x}$.

There are no explicit parentheses here, but the square root acts like parentheses, and it designates an inside function of $g = x^2 + 2e^x$. The outside function is therefore $f = \sqrt[4]{x}$. If we rewrite $\sqrt[4]{x}$ as $x^{1/4}$, we can use the power rule and find $f' = \frac{1}{4}x^{-3/4}$. We also have $g' = 2x + 2e^x$. Hence, the chain rule gives

$$\begin{aligned}\frac{d}{dx}\sqrt[4]{x^2+2e^x} &= f'(g)\cdot g' \\ &= \frac{1}{4}(x^2+2e^x)^{-3/4}(2x+2e^x) \\ &= \frac{2x+2e^x}{4}\frac{1}{\sqrt[4]{x^2+2e^x}^3} \\ &= \boxed{\frac{x+e^x}{2\sqrt[4]{x^2+2e^x}^3}}.\end{aligned}$$

That's as simplified as we can get the answer to be.

One more quick example.

Example **Proof of a^x rule**

Problem Prove the rule $\frac{d}{dx}\ a^x = \ln(a)\cdot a^x$.

Recall that $e^{\ln(a)} = a$. To prove this rule, we rewrite $a^x = (e^{\ln(a)})^x = e^{x\ln(a)}$. We are then computing

$$\frac{d}{dx}\ a^x = \frac{d}{dx}\ e^{x\ln(a)}$$

To compute this derivative, we set $f = e^x$ and $g = x\ln(a)$. We find $f' = e^x$ and $g' = \ln(a)$, so by the chain rule

$$\begin{aligned}\frac{d}{dx}\ a^x &= \frac{d}{dx}\ e^{x\ln(a)}\\ &= f'(g)\cdot g'\\ &= e^{x\ln(a)}\cdot\ln(a)\end{aligned}$$

We've already shown that $a^x = e^{x\ln(a)}$, so this simplifies to $a^x \cdot \ln(a)$, as desired.

Chapter 30

HOMEWORK: CHAIN RULE

1. Watch this video from Khan Academy:
   Chain Rule Definition and example
2. Take the derivative of the following functions, each of which involves the chain rule.

   a. $a(x) = (x^2 + 5)^{20}$
      $40x(x^2 + 5)^{19}$
      ans
   b. $b(x) = e^{x^2}$
      $2xe^{x^2}$
      ans
   c. $c(x) = (kx + r)^n$ for constants k, r, n.
      $kn(kx + r)^{n-1}$
      ans
   d. $d(x) = (\ln(x))^3 + \ln(x^3)$
      $\frac{3(\ln(x))^2}{x} + \frac{3}{x}$
      ans
   e. $e(x) = \sin(\cos(x))$
      $-\sin(x)\cos(\cos(x))$
      ans
   f. $f(x) = e^{\sin(x)+\cos(x)}$
      $(\cos(x) - \sin(x))e^{\sin(x)+\cos(x)}$
      ans
   g. $g(x) = \sqrt{3x^2 - 5x + 6}$
      $\frac{6x-5}{2\sqrt{3x^2-5x+6}}$
      ans
   h. $h(x) = e^{-x}$
      $-e^{-x}$
      ans

3. For each problem, try simplifying the logarithm first, then taking the derivative.

   a. $\frac{d}{dx} \ln(x^3)$
      $\frac{3}{x}$
      ans

b. $\frac{d}{dx}\ln(xe^x)$
$\frac{1}{x}+1$
ans

4. Use logarithm rules to explain why $\frac{d}{dx}\ln(e^5\cdot x)=\frac{d}{dx}\ln(x)$.
Using logarithm rules, we have that $ln(e^5\cdot x)=\ln(x)+\ln(e^5)=\ln(x)+5$. This has the same derivative as $\ln(x)$ since we are just adding a constant.
ans
5. Recall that $\ln(x)$ and e^x are inverse functions. This means that $\ln(e^x)=x$, and $e^{\ln(x)}=x$ (that is, the e and the $\ln$ cancel out if you do one right after the other). This fact allows us to compute $\frac{d}{dx}2^x$.

a. Simplify $e^{\ln(2)}$
$=2$
ans
b. Simplify $\ln(e^2)$.
$=2$
ans
c. Simplify $e^{\ln(2)+x}$
$2e^x$
ans
d. Simplify $(e^{\ln(2)x})$
2^x
ans

e. Use part (d) to compute $\frac{d}{dx}2^x$.
$\ln(2)2^x$
ans

CHAPTER 31

MULTIRULE DERIVATIVES

Okay, let's talk about $\frac{d}{dx}\ e^{x^2+x}\sin(x)$. If you're thinking this looks like a product rule, but it also looks like a chain rule, you're right. To compute this derivative, we need to do the chain rule and the product rule. This is because it is a multirule problem. Let's do this example

Example **Multirule**

Problem Compute $\frac{d}{dx}\ e^{x^2+x}\sin(x)$.

The way I like to break this down is to consider a little rule and a big rule. In this case, the little rule is the chain rule problem $\frac{d}{dx}\ e^{x^2+x}$. If we do this problem, we see that $f = e^x$, $f' = e^x$, $g = x^2 + x$ and $g' = 2x + 1$. So we have

$$\frac{d}{dx}\ e^{x^2+x} = e^{x^2+x}(2x+1). \tag{1}$$

Now we are ready to do the big rule, which is the product rule. **At this point we go back to the original problem** $\frac{d}{dx}\ e^{x^2+x}\sin(x)$. For this product rule, we see $f = e^{x^2+x}$, $g = \sin(x)$, $g' = \cos(x)$. What is f'? Why, that's what we just computed in the equation above! So $f' = e^{x^2+x}(2x+1)$. Putting this all together with the product rule $fg' + gf'$, we have

$$\begin{aligned}\frac{d}{dx}\ e^{x^2+x}\sin(x) &= fg' + gf' \\ &= \boxed{e^{x^2+x}\cos(x) + \sin(x)e^{x^2+x}(2x+1)}.\end{aligned}$$

Example **Multirule**

Problem Compute $\frac{d}{dx}\frac{x}{\sin(x^2+x)}$.

little chain rule: $\frac{d}{dx}\sin(x^2+x)$

$$\begin{array}{ll} f = \sin(x) & g = x^2 + x \\ f' = \cos(x) & g' = 2x + 1 \end{array}$$

Result: $\cos(x^2+x)\cdot(2x+1)$

Big quotient rule (aka the whole problem): $\frac{d}{dx}\frac{x}{\sin(x^2+x)}$

$$f = x \quad g = \sin(x^2+x)$$
$$f' = 1 \quad g' = \cos(x^2+x)\cdot(2x+1)$$

Result: $\boxed{\dfrac{\sin(x^2+x)\cdot 1 - x\cos(x^2+x)\cdot(2x+1)}{(\sin(x^2+x))^2}}$

CHAPTER 32

HOMEWORK: MULTIRULE DERIVATIVES

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1. Each of the problems below involves a combination of two of the following: product rule, quotient rule, chain rule. Give them a shot!
   Answer key note: the answers below are simplified, in some cases more so than I'd expect you to on a quiz or exam, but it is still good practice to try to simplify and see if you got the same thing I did.
   ans

   a. $\frac{d}{dx}\ x^2 \ln(x) e^x$
   $x^2 \ln(x) e^x + 2x \ln(x) e^x + x e^x$
   ans
   b. $\frac{d}{dx}\ (2x+1)^5 (3x-1)$
   $3(2x+1)^5 + 10(3x-1)(2x+1)^4$
   ans
   c. $\frac{d}{dx}\ \sqrt{\sin(x^2)}$
   $\frac{1}{2}(\sin(x^2))^{-1/2} \cos(x^2) 2x$
   ans
   d. $\frac{d}{dx}\ e^{1/(x^2-1)}$
   $e^{\frac{1}{x^2-1}} \cdot \frac{-2x}{(x^2-1)^2}$
   ans
   e. $\frac{d}{dx}\ \frac{x\ln(x)}{x+1}$
   $\frac{\ln(x)+x+1}{(x+1)^2}$
   ans
   f. $\frac{d}{dx}\ \dfrac{\dfrac{1}{\ln(x)} + \ln(x)}{\ln(x)}$
   $\frac{-2}{x(\ln(x))^3}$
   ans
   g. $\frac{d}{dx}\ e^{\cos(x)\sin(x)}$
   $e^{\cos(x)\sin(x)} \cdot ((\cos(x))^2 - (\sin(x))^2)$
   ans
   h. $\frac{d}{dx}\ \sin\left(\frac{x\ln x}{e^x}\right)$
   $\cos\left(\frac{x\ln x}{e^x}\right) \cdot \frac{\ln(x)+1-x\ln(x)}{e^x}$

ans

Chapter 33

ANTI-DERIVATIVES

We'll see in a future chapter that we will need to be able to be able to *undo* a derivative. That is, given the answer of a derivative problem, what is the original question?

Given a function $f(x)$, $F(x)$ is the *anti-derivative* of $f(x)$ if $F'(x) = f(x)$. We denote this by

$$\int f(x)\ dx = F(x).$$

For now, just consider $\int$ and dx to be notation that tells you to take an anti-derivative.

Let us look at some examples with the power rule. The following is a table of derivative problems.

$F(x)$	$F'(x) = f(x)$
1	0
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$

Now let's look at anti-derivatives. All we do is what was the original problem is now the answer, and what was the answer is now the original problem.

$f(x)$	$\int f(x)\ dx = F(x)$
0	1
1	x
$2x$	x^2
$3x^2$	x^3
$4x^3$	x^4

There is one twist we have to watch out for: anti-derivatives are not unique. Consider the following derivatives:

$F(x)$	$F'(x) = f(x)$
7	0
$x + 7$	1
$x^2 + 7$	$2x$
$x^3 + 7$	$3x^2$
$x^4 + 7$	$4x^3$

This would lead us to the following anti-derivatives

$f(x)$	$\int f(x)\ dx = F(x)$
0	7
1	$x+7$
$2x$	x^2+7
$3x^2$	x^3+7
$4x^3$	x^4+7

So, suppose I ask the question "what is the anti-derivative of $2x$?" There are many possible solutions, including x^2, x^2+7, x^2+42, etc. However, all of these solutions are basically the same thing. To get around this problem, we say often use the notation x^2+C. Here, C is called the *constant of integration*. In this form, the antiderivative is called an *indefinite integral*.

If we are dealing with a polynomial, we know the derivative follows the power rule. With anti-derivatives, it follows the *inverse power rule*:

$$\boxed{\int x^m\ dx = \frac{x^{m+1}}{m+1} + C}$$

Note that this only works if $m \neq -1$. Can you see what goes wrong in the formula if $m = -1$?

We also have the rules of linearity work in reverse. So when taking anti-derivatives, you can just look at one term at a time, and the constant will stay during the anti-derivative process.

$$\boxed{\int cf\ dx = c\int f\ dx}$$

$$\boxed{\int f+g\ dx = \int f\ dx + \int g\ dx}$$

Here are some examples.

Example **Antiderivative Examples**

- **Problem** Find $\int x^7\ dx$.

Using the inverse power rule, we have $\int x^7\ dx = \boxed{\frac{x^8}{8}+C}$.

- **Problem** Find $\int 6x^2\ dx$.

The constant will "come along for the ride", or not be affected by the anti-derivative process. Hence we see

$$\begin{aligned}\int 6x^2\ dx &= 6\int x^2\ dx\\ &= 6\left(\frac{x^3}{3}+C\right)\\ &= 6\frac{x^3}{3}+6C\\ &= 2x^3+6C.\end{aligned}$$

Note that the 6 in the $6C$ part of the answer is not really necessary. What is important is we are indicating that you can add any constant of integration we'd like. Therefore, we will often replace the $6C = D$, and we have the answer

$$\int 6x^2\ dx = 2x^3 + D.$$

In fact, often we will just wait to add the C until the end. In that case, we would arrive at an answer of $2x^3$, then we'd add the C at that stage, and get an answer of

$$\int 6x^2\ dx = \boxed{2x^3 + C}.$$

- Problem Find $\int \frac{1}{2}x^3\ dx$

The $\frac{1}{2}$ is a constant that does not affect the integration, so we see

$$\begin{aligned}\int \frac{1}{2}x^3\ dx &= \frac{1}{2}\int x^3\ dx\\ &= \frac{1}{2}\left(\frac{x^4}{4}\right)\\ &= \boxed{\frac{x^4}{8}+C}.\end{aligned}$$

Notice we didn't add the constant of integration until the last step, and that's perfectly okay.

- Problem Find $\int 6x^5 + 6x\ dx$

In a sum, we can just treat each term separately. And the constants come along for the ride.

$$\begin{aligned}\int 6x^5 + 6x\ dx &= 6\int x^5\ dx + 6\int x\ dx \\ &= 6\left(\frac{x^6}{6}\right) + 6\left(\frac{x^2}{2}\right) \\ &= \boxed{x^6 + 3x^2 + C}.\end{aligned}$$

- Problem Find $\int 4e^x + \frac{3}{x} + \frac{\sin(x)}{5}\ dx$.

We treat each term separately.

- $\int 4e^x\ dx$. In this term, the 4 is a constant multiple, and the e^x doesn't change, so we get
$$\int 4e^x\ dx = 4e^x.$$
- $\int \frac{3}{x}\ dx$. Okay, a bit harder now. Notice the 3 is a constant multiple, so let's take that out: $3\int \frac{1}{x}\ dx$. We recognize this as the derivative of $\ln(x)$, so the answer is
$$\int \frac{3}{x}\ dx = 3\ln(x).$$
- $\int \frac{\sin(x)}{5}\ dx$. For this one, the division by 5 is the same a constant multiple of $\frac{1}{5}$. We bring this outside and get: $\frac{1}{5}\int \sin(x)\ dx$. We know that $\frac{d}{dx}\cos(x) = -\sin(x)$, so we know $\frac{d}{dx} - \cos(x) = \sin(x)$. This gives the final answer of
$$\int \frac{\sin(x)}{5}\ dx = -\frac{1}{5}\cos(x).$$

Putting it altogether, we get a final answer of $\boxed{4e^x + 3\ln(x) - \frac{1}{5}\cos(x) + C}$. Nice work!

Chapter 34

HOMEWORK: ANTI-DERIVATIVES

1. Compute the following indefinite integrals. Don't forget the constant of integration!

   a. $\int x^2 + x \ dx$
   $\frac{1}{3}x^3 + \frac{1}{2}x^2 + C$
   ans

   b. $\int \sqrt{x} + \sqrt[3]{x} \ dx$
   $\frac{2}{3}x^{3/2} + \frac{3}{4}x^{4/3} + C$
   ans

   c. $\int \frac{2}{x} \ dx$
   $2\ln(x)$ Wait! I mean $2\ln(x) + C$. Oops!
   ans

   d. $\int (3x+1)^2 \ dx$
   $3x^3 + 3x^2 + x + C$
   ans

   e. $\int \frac{x+1}{x} \ dx$
   $\frac{1}{2}x^2 + \ln(x) + C$
   ans

   f. $\int 3x^2 + 2x + 1 \ dx$
   $x^3 + x^2 + x + C$
   ans

PART IV

MORE DERIVATIVE INTUITION

CHAPTER 35

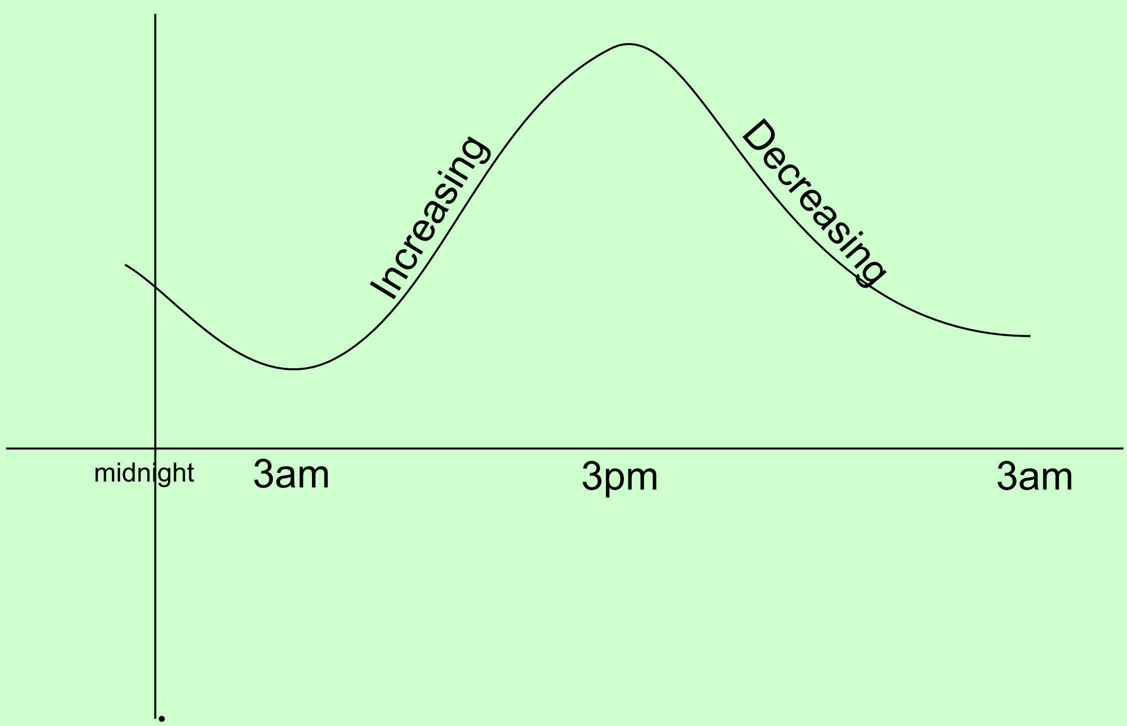
DERIVATIVES AND GRAPHS

As we've seen, one of the most important connections between a function and its derivative is that *a positive derivative means the quantity is increasing, and a negative derivative means the quantity is decreasing.*

Example **Increasing and Decreasing**

- Problem Outside temperature has a positive derivative from 3am to 3pm, and a negative derivative from 3pm to 3am. Draw a graph of this, and label each part of the graph as "increasing" or "decreasing".

With the positive derivative from 3am to 3pm, this should go up and be labeled "increasing". From 3pm to 3am, the graph is going down and labeled "decreasing".



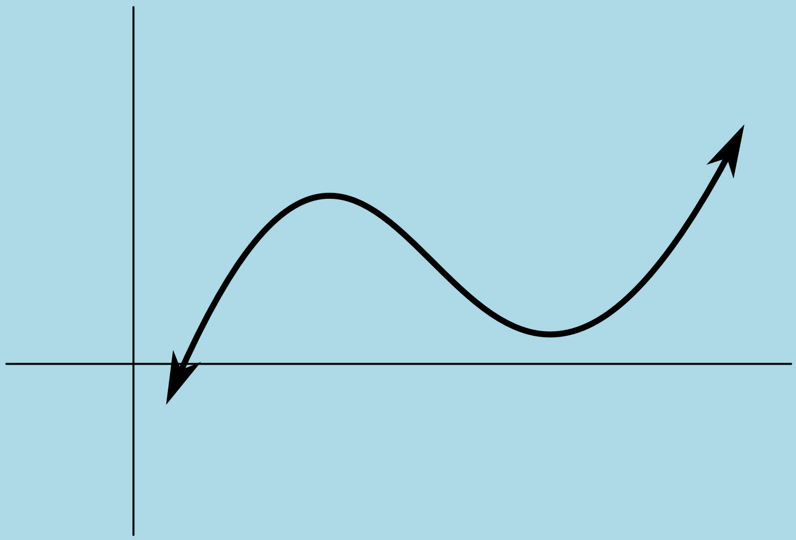
At the interface between increasing and decreasing, at 3pm, is when the temperature is the highest.

This is the key to one of the most useful applications of calculus: optimization! *Optimization* is either finding out when a quantity is maximized, or as high as possible, or finding out when a quantity is minimized, or as low as possible. Often this is at the interface of increasing and decreasing, and thus at the where a function goes from positive derivative to negative derivative. Hence, one of the most important maxims of calculus: *optimization happens when the derivative is zero!* We will come back to this in a future section.

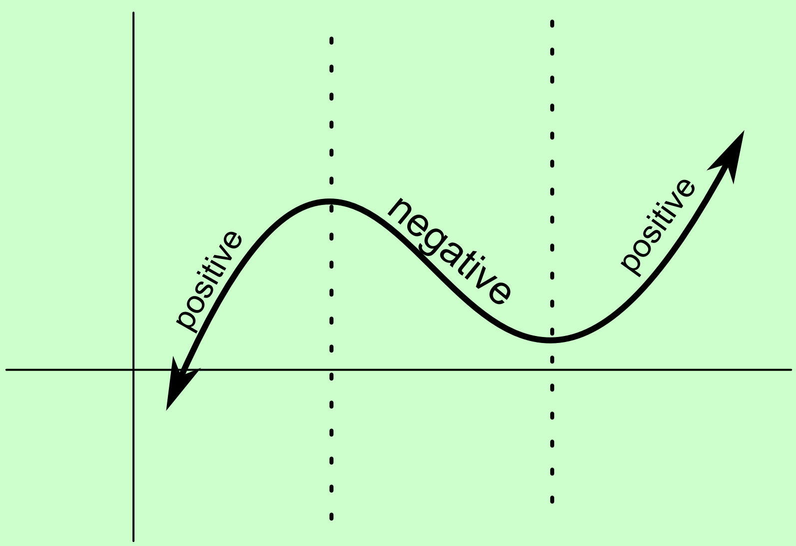
For now, using this idea of when the derivative is positive, negative, or zero, we can draw a rough sketch of the derivative based on the graph of a function. Let's see an example

Example **Derivative sketching**

Problem Sketch the derivative of the following function.

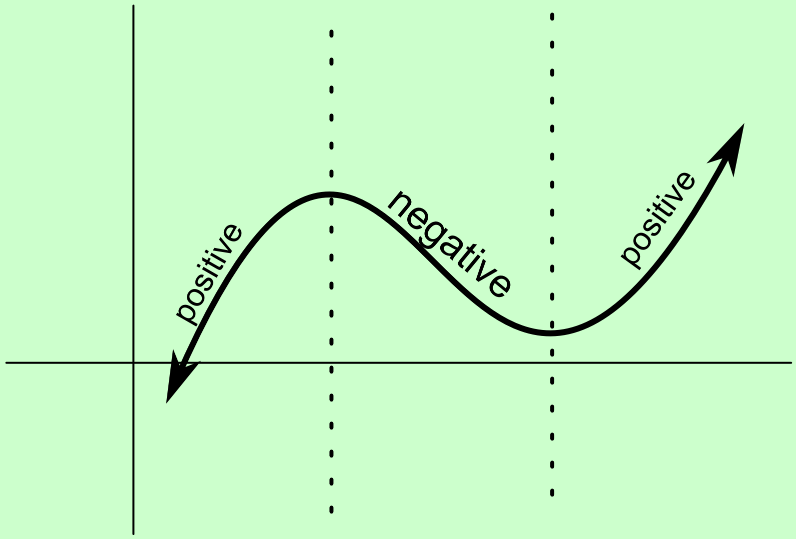


When sketching the derivative, keep this idea in mind: *slopes become y-values*. First, let's mark where the derivative is zero:

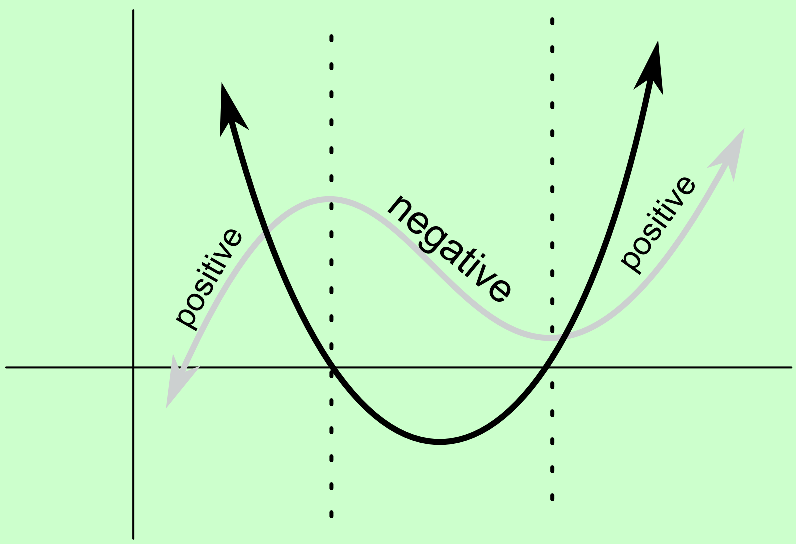


These are the places that, in the derivative graph, have zero for the y-value. That means these are the x-intercepts!

Now let's mark where the derivative is positive, and where it is negative.



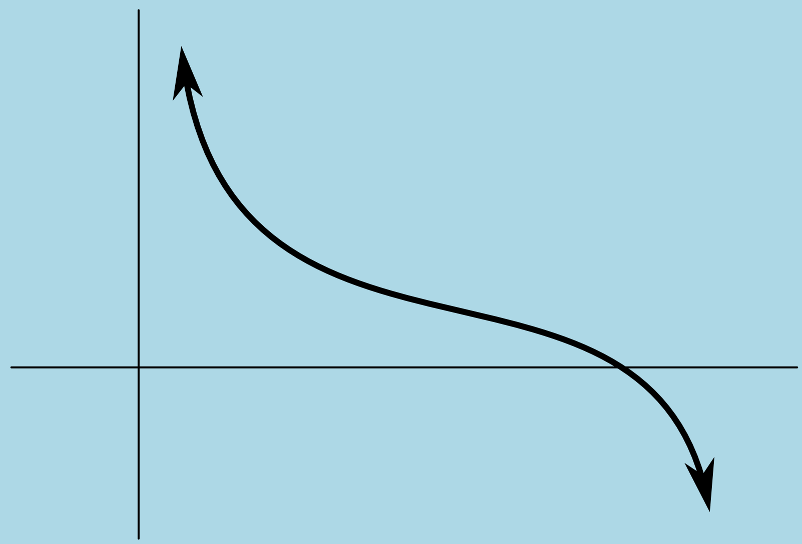
Finally, we can use this as a rough guide for a sketch, again keeping in mind slope becomes y -values. Here, the derivative is in black, while the original function is in grey.



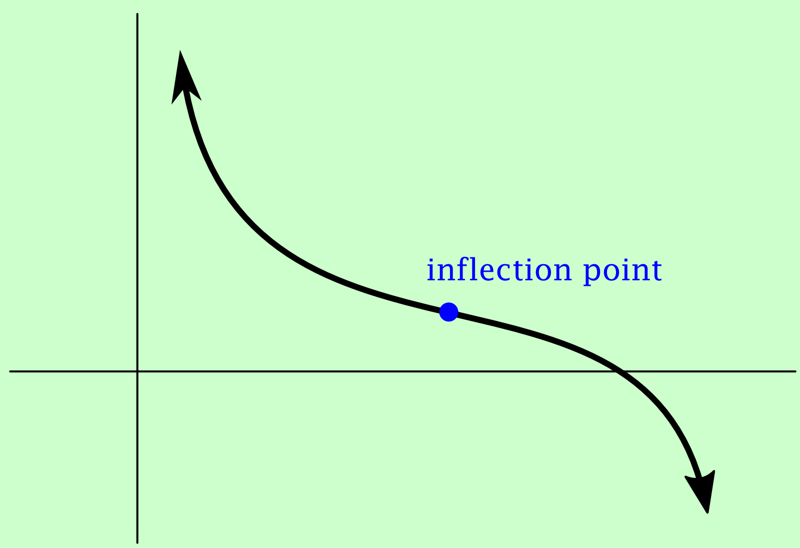
Let's see another example.

Example **Derivative Sketching 2**

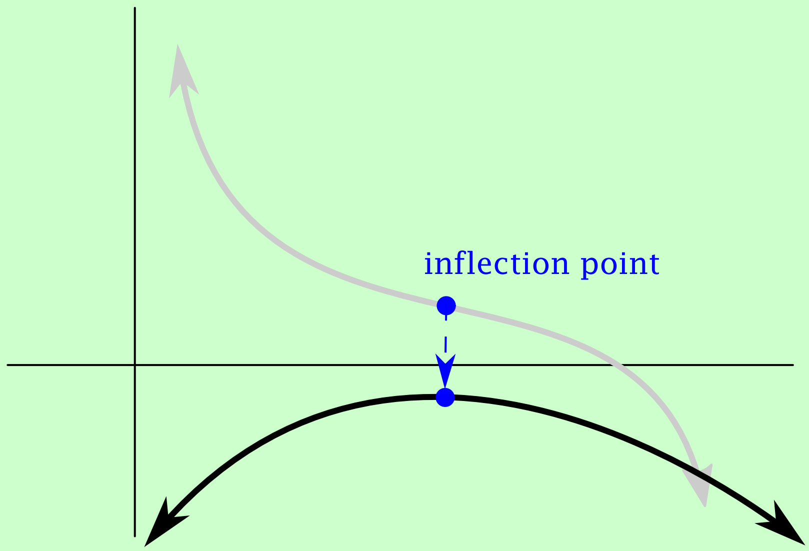
Problem Sketch the derivative



In this case, the graph is always going down, so the derivative is always negative. That doesn't really tell you a lot about what the derivative graph looks like. However, there is a special point called an *inflection point* right here:



It's not where the slope is zero, but it is where the slope gets the closest to zero. Everywhere else the slope is more negative than at the inflection point. So in the derivative graph this becomes a maximum or highest point, like this:

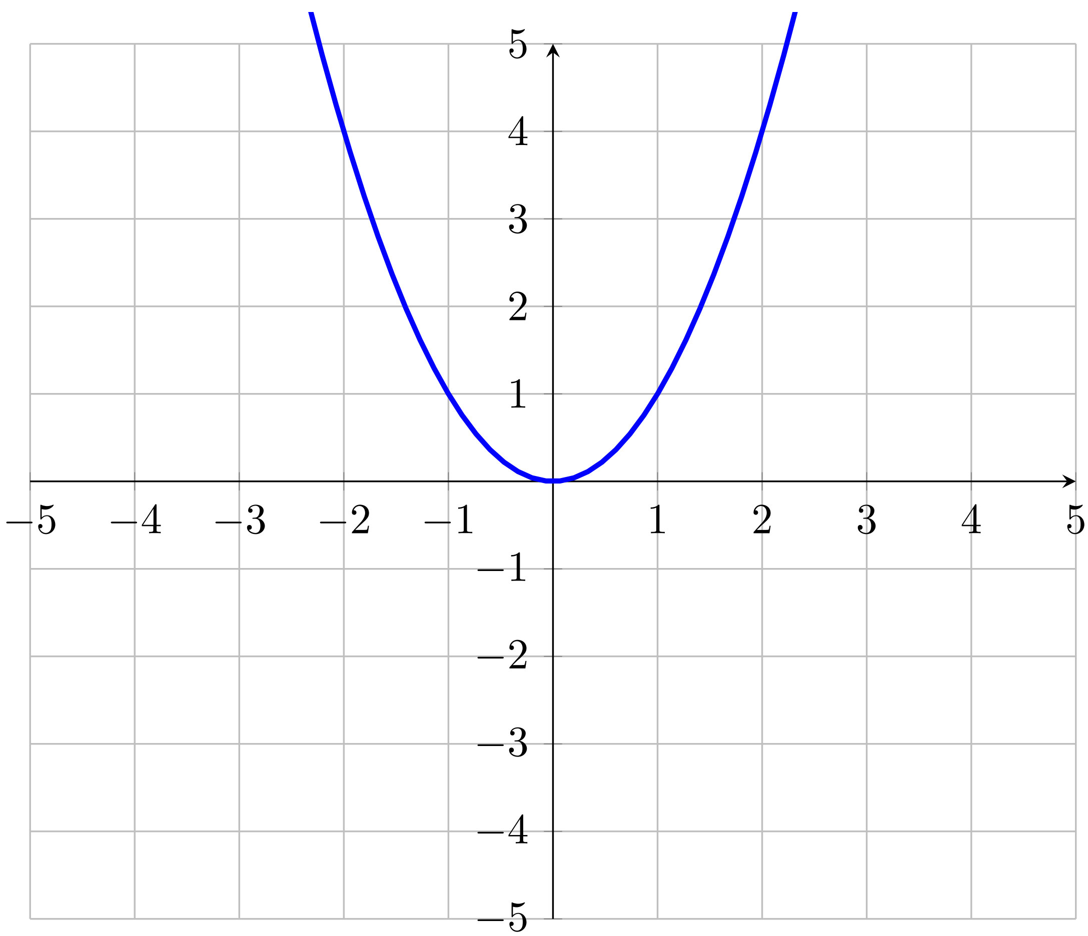


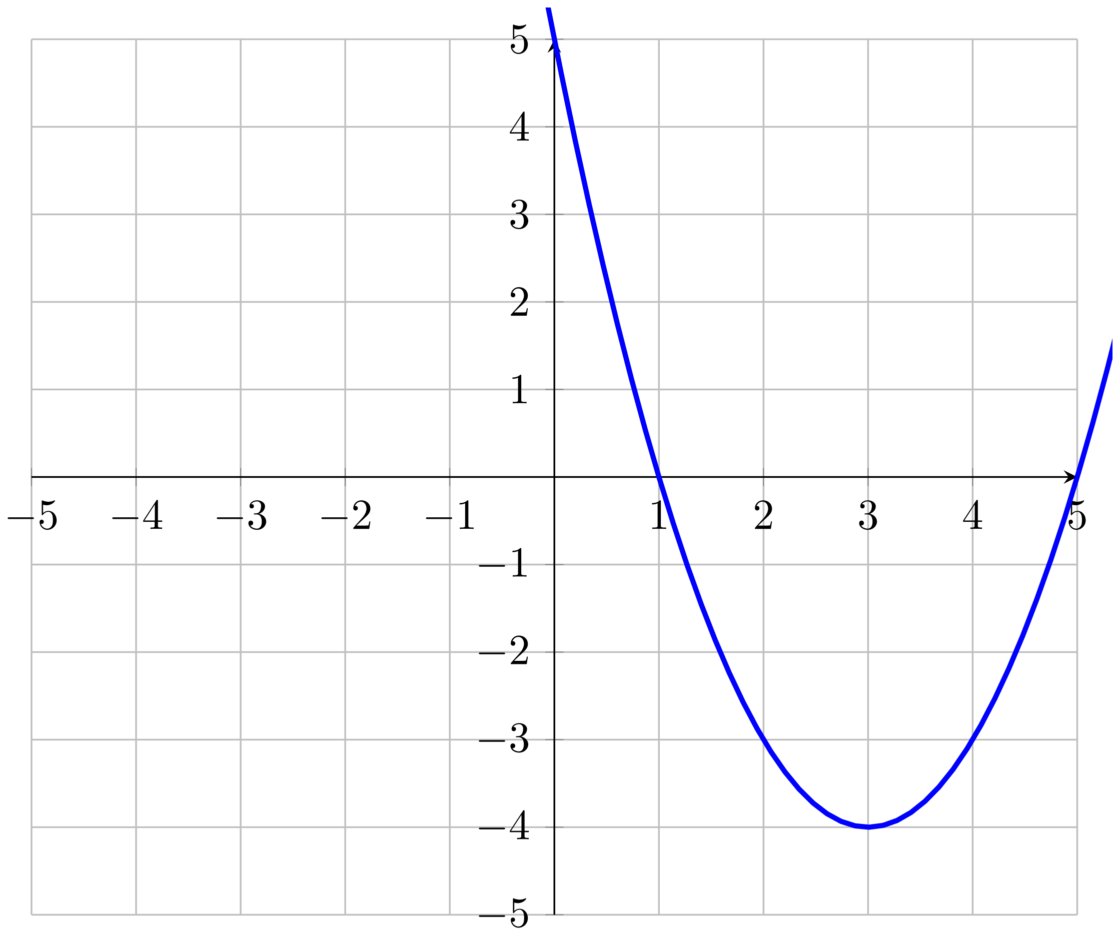
As long as that inflection point is the highest point (but still a negative y-value due to the negative slope of the original graph), that's about the best you can do on that one.

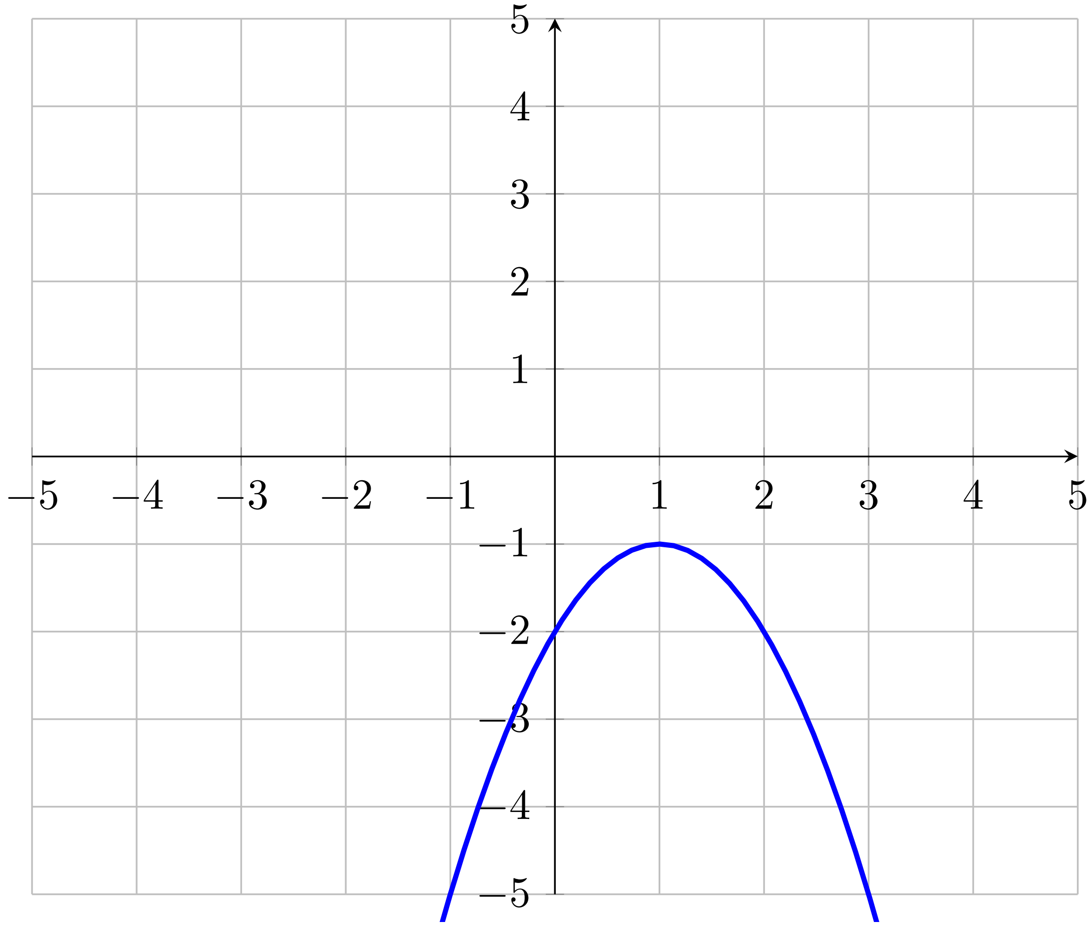
CHAPTER 36

HOMEWORK: DERIVATIVE GRAPHS

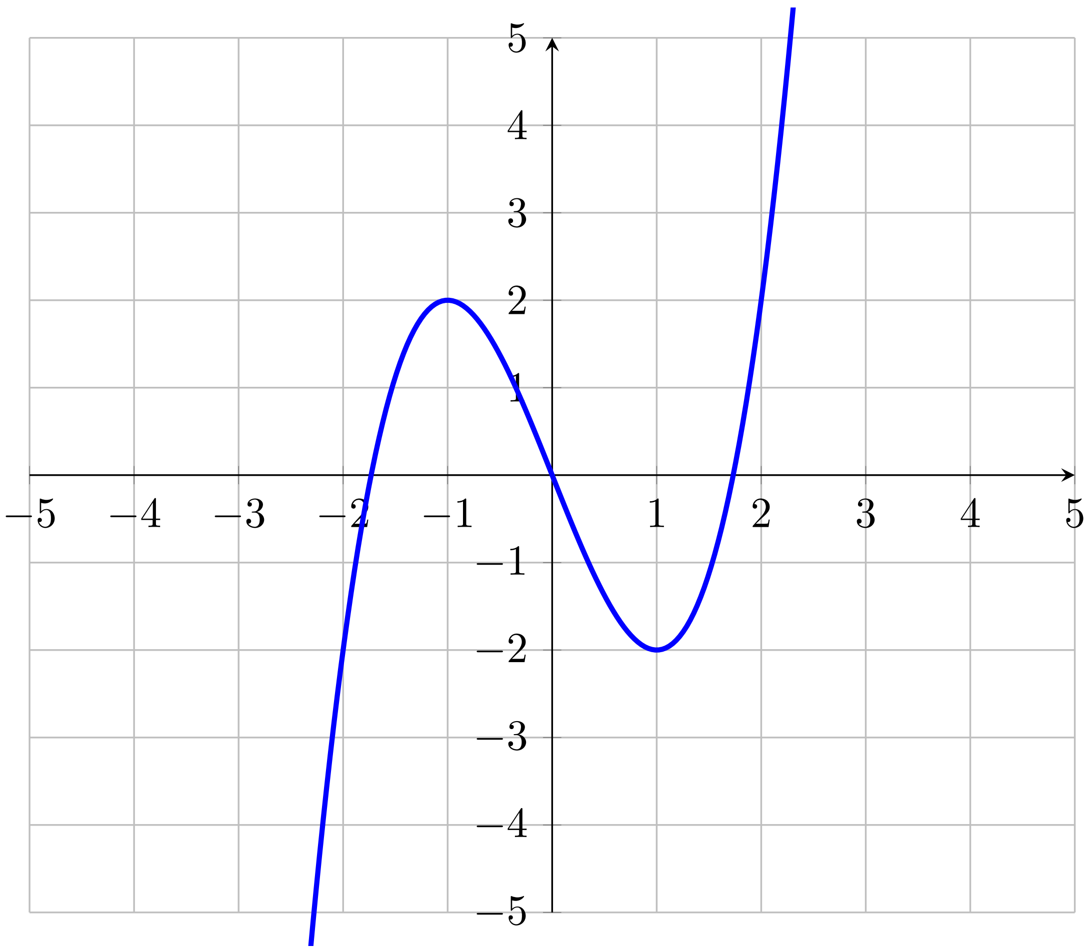
1. For each graph, sketch the derivative graph (you can sketch the derivative on the same axes as the problem if you prefer)

a. 

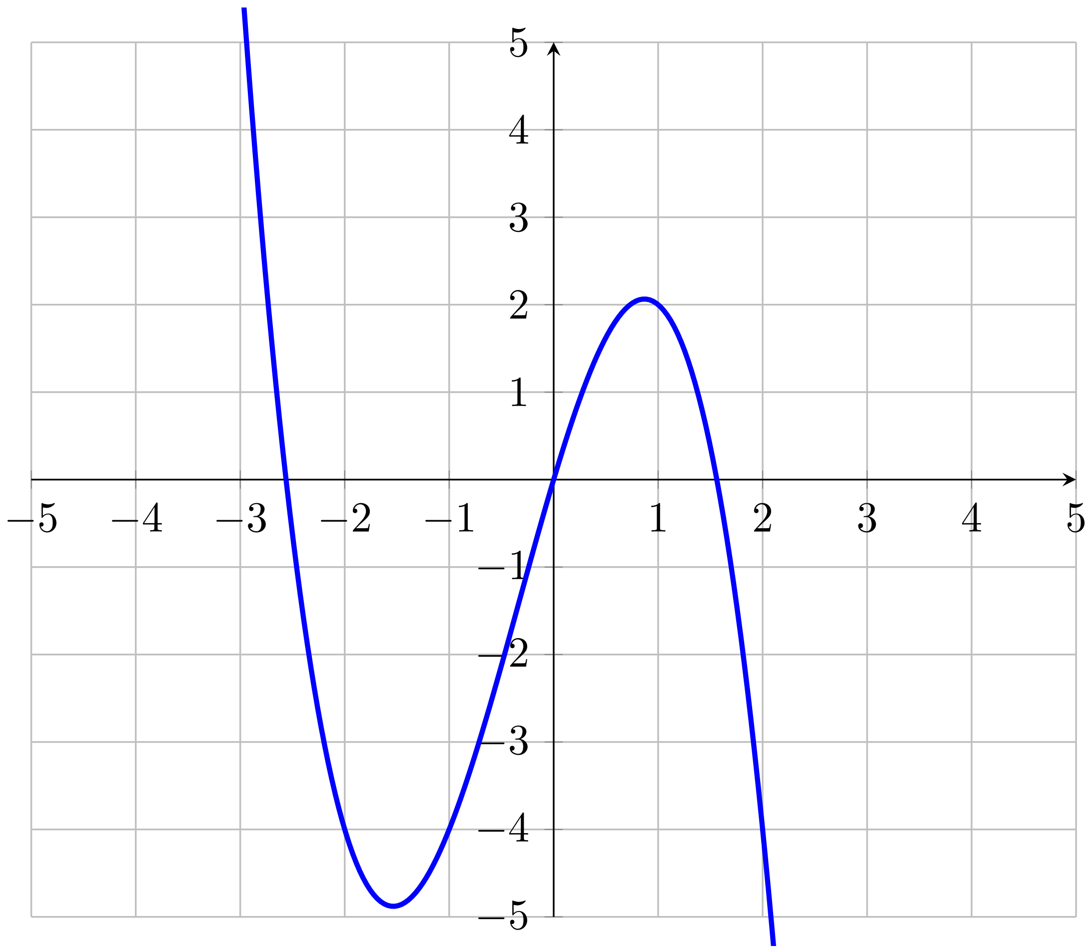
b. 

c. 

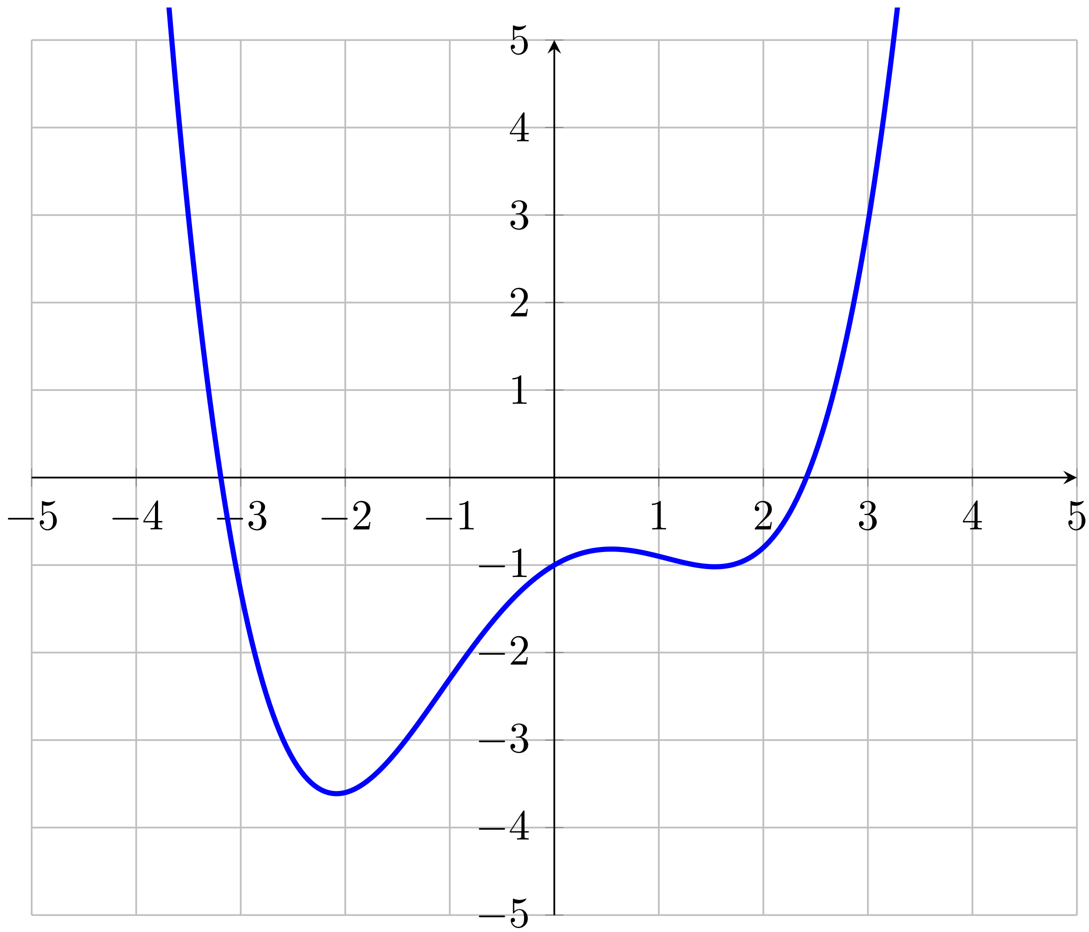
d.

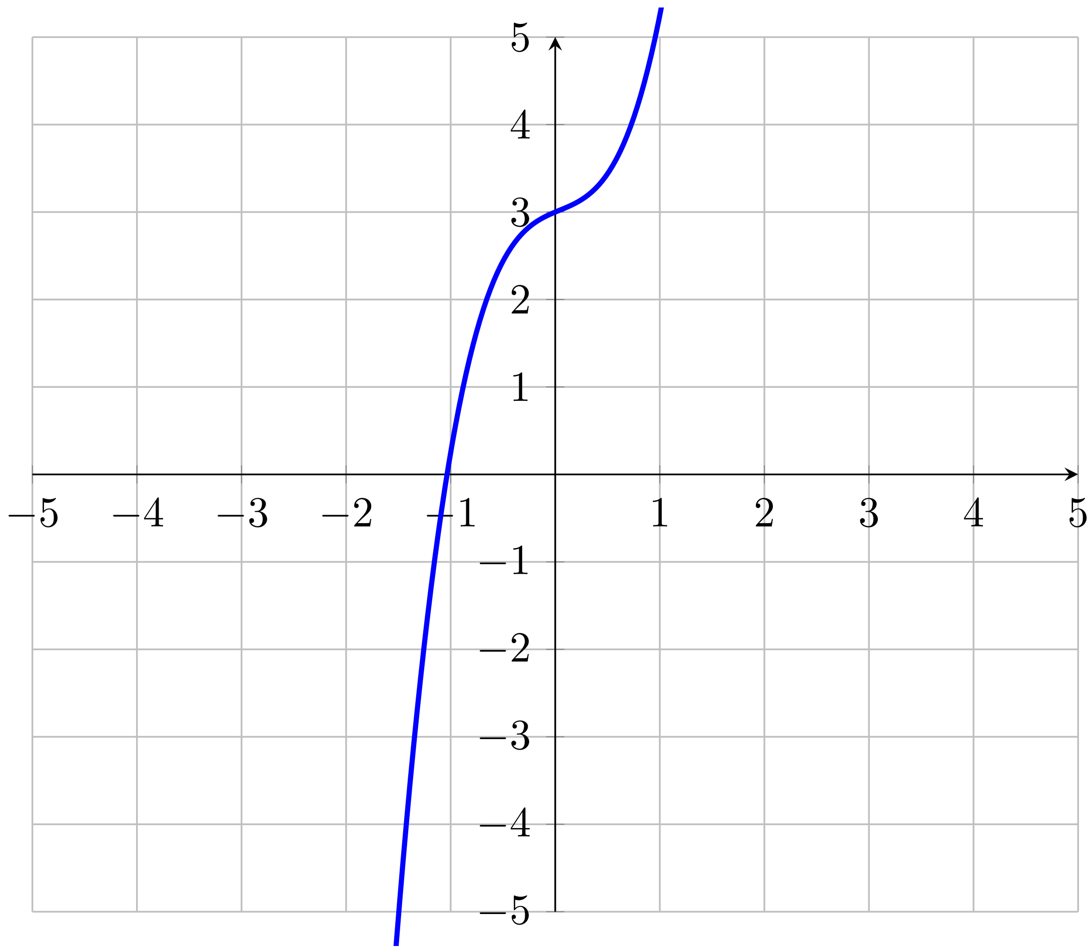


e.



f.



g. 

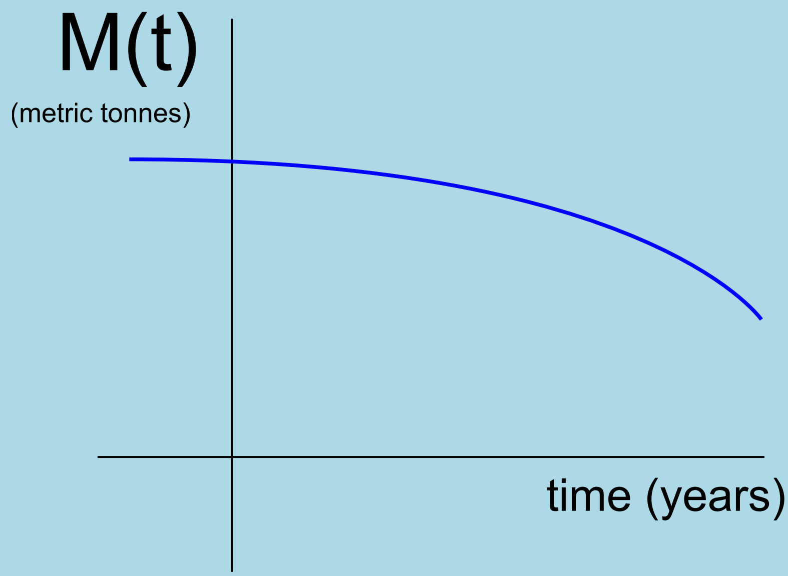
Chapter 37

SECOND DERIVATIVES AND INTERPRETING THE DERIVATIVE

We understand velocity versus position at this point, but what about the derivative of other changing quantities? What does it mean? Let's look at an example.

Example **Glacial Loss**

Problem Let $M(t)$ be the mass in metric tonnes of a glacier over a given time t, where t is measured in years. What do you think the graph of $M(t)$ looks like? Given global warming, it's probably going down, like this:



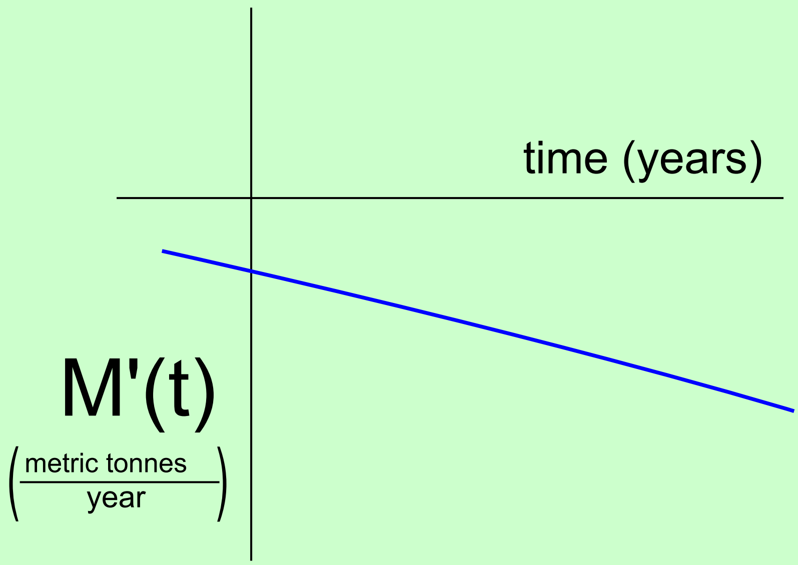
Question: What is the derivative measuring now? What would the graph of the derivative look like?

Answers: Good questions! Just like the derivative of position is velocity, or how fast the position is changing, the derivative of $M(t)$ is going to be how fast the mass of the glacier is changing. In this case, it is how fast it is melting.

We can automatically assign units to the derivative. Since the original graph is metric tonnes on the y-axis, and years on the x-axis, we know the unit of the derivative (unless we want to convert) is going to be metric tonnes per year. We might then talk about the derivative being $-15\ \frac{\text{metric tonnes}}{\text{year}}$.

Notice I assumed the derivative was negative. Why did I do that? That is because the graph is going down, and the glacier is losing mass.

Note further that it isn't just going down like a line going down. It's a curve downward. What does this mean for the derivative? Well, the derivative starts negative since it's going down, and continues to be negative as it goes down. But it goes down faster and faster as you move to the right — that means the derivative is getting more and more negative. Like this:



So we can see that this derivative is negative, but it's worse than that — it's negative and going down. That's not good news for the glacier. Looking at whether the derivative is going up or down is known as the *second derivative*. We will see in the next section this is easy to calculate.

As we saw in the above example, sometimes we need to repeat the process of taking derivatives. This gives the second derivative, third derivative, and so on. The notation is

$$f''(x), \quad \text{or} \quad \tfrac{d^2}{dx^2}f$$

You can also take three, four, or more derivatives. Instead of writing several primes, we write $f^{(4)}$ for the fourth derivative, $f^{(5)}$ for the fifth derivative, and so on. Let's do a couple of examples.

Example **Multiple Derivatives**

Problem Let $f(x) = x^3 + x^2$. Find

1. $f'(x)$
2. $f''(x)$
3. $f'''(x)$
4. $f^{(4)}(x)$

For (1), we use the power rule and see that $f'(x) = \boxed{3x^2 + 2x}$.

For (2), we use the power rule again applied to $3x^2 + 2x$. So we have

$$f''(x) = 3\tfrac{d}{dx}x^2 + 2\tfrac{d}{dx}x = 3(2x) + 2(1) = \boxed{6x+2}$$

For (3), we take the derivative of $6x + 2$. This is $\boxed{6}$.

For (4), we take the derivative of 6 and that is $\boxed{0}$.

Example **Multiple Derivatives again**

Problem Find $\frac{d^2}{dx^2}\ln(x)$.

To get a second derivative, we first need to take one derivative. We see the derivative of $\ln(x)$ is $\frac{1}{x}$. We then take the derivative again, and we see

$$\begin{aligned}
\frac{d^2}{dx^2}\ln(x) &= \frac{d}{dx}\frac{1}{x} \\
&= \frac{d}{dx}x^{-1} \\
&= -1x^{-2} \\
&= \boxed{-\frac{1}{x^2}}
\end{aligned}$$

Example **More derivatives with** $\cos(x)$

Problem Find the first four derivatives of $f(x) = \cos(x)$.

Here, we see that $f'(x) = -\sin(x)$, $f''(x) = -\cos(x)$, $f'''(x) = \sin(x)$, and $f^{(4)}(x) = \cos(x)$. Hence, we start repeating answers after we take four derivatives!

CHAPTER 38

HOMEWORK: SECOND DERIVATIVES AND INTERPRETING THE DERIVATIVE

---

1. Given each $f(t)$, describe in one sentence the meaning of $f'(t)$.

   a. Let $f(t)$ be the distance (in miles) an astronaut is from the surface of the earth as he blasts off towards space. Here t is measured in hours.
   $f'(t)$ is the speed of the astronaut in miles per hour
   ans

   b. Let $f(t)$ be the number of gallons of diesel gasoline in the tank of a truck, with t measured in hours.
   $f'(t)$ is how fast fuel is being burned, in gallons per minute. It could also represent how fast the fuel is being filled up at a gas station.
   ans

   c. Let $f(t)$ be the concentration of NaCl in parts per million within the cytoplasm of a cell. Here, t is measured in minutes.
   $f'(t)$ is the rate the concentration of NaCl is increasing in parts per million per minute.
   ans

   d. Let $f(t)$ be the speed of a runner (in feet per second), and let t be measured in seconds.
   $f'(t)$ is the acceleration of the runner, or the rate at which the runner's speed is increasing in (feet per second) per second.
   ans

   e. Let $f(t)$ be the rate (in dollars per hour) that you are paid, where t is measured in months.
   $f'(t)$ is like how fast are you getting raises, measured in (dollars per hour) per month.
   ans

2. For each of the functions below, compute the derivative twice. That is, compute $f'(t)$, then take the derivative of $f'(t)$ to find $f''(t)$.

   a. $f(t) = 5t^2 - 6t + 2$
   $10t - 6, 10$
   ans

   b. $f(t) = t^3 + e^t - \sqrt{t}$.
   $3t^2 + e^t - 1/2t^{-1/2}, 6t + e^t + 1/4t^{-3/2}$

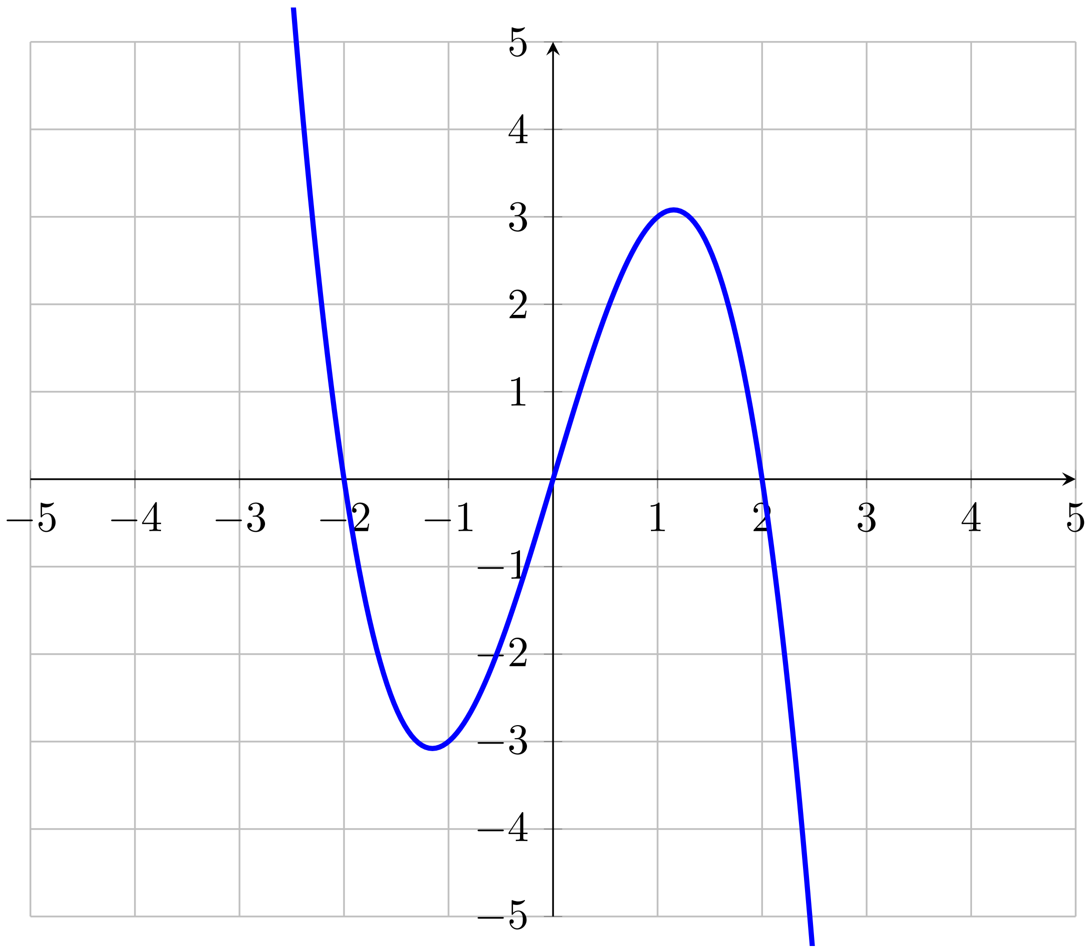
ans

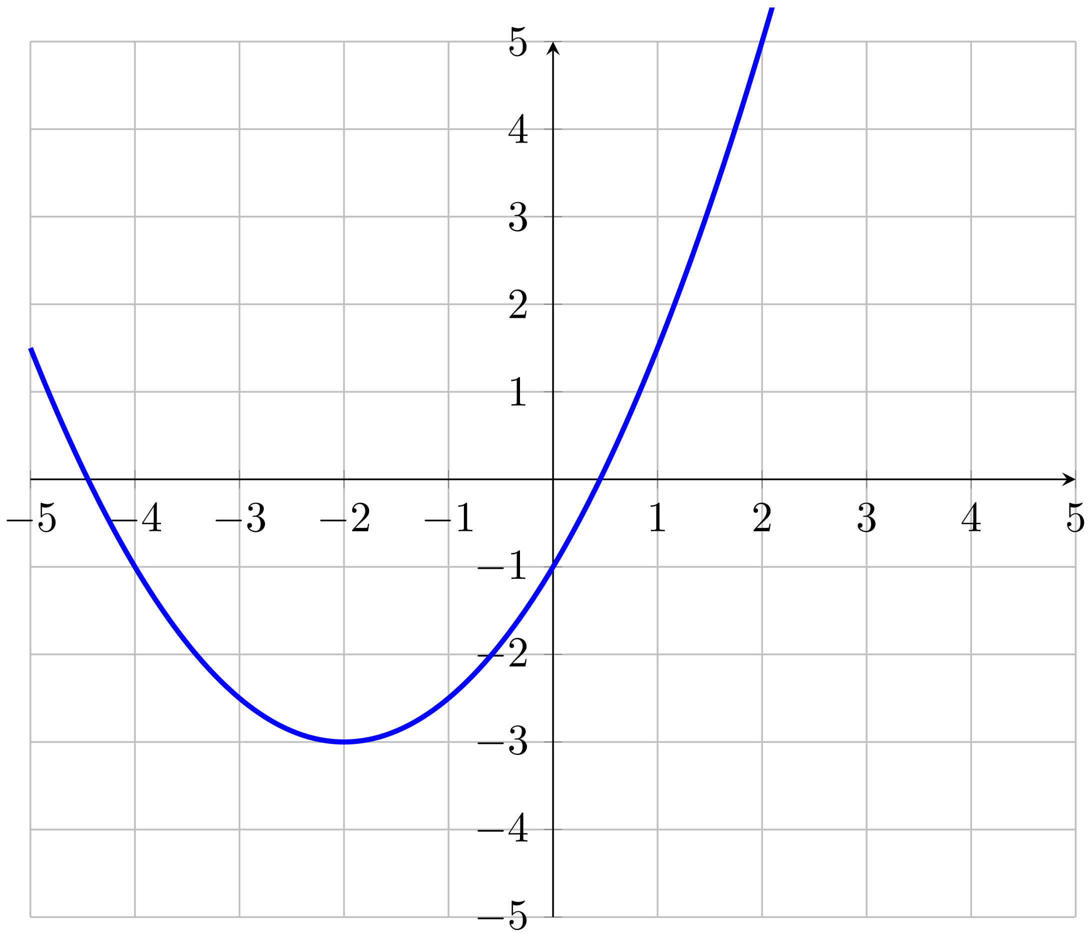
c. $f(t) = \frac{1}{t}$
$-t^{-2}, 2t^{-3}$
ans

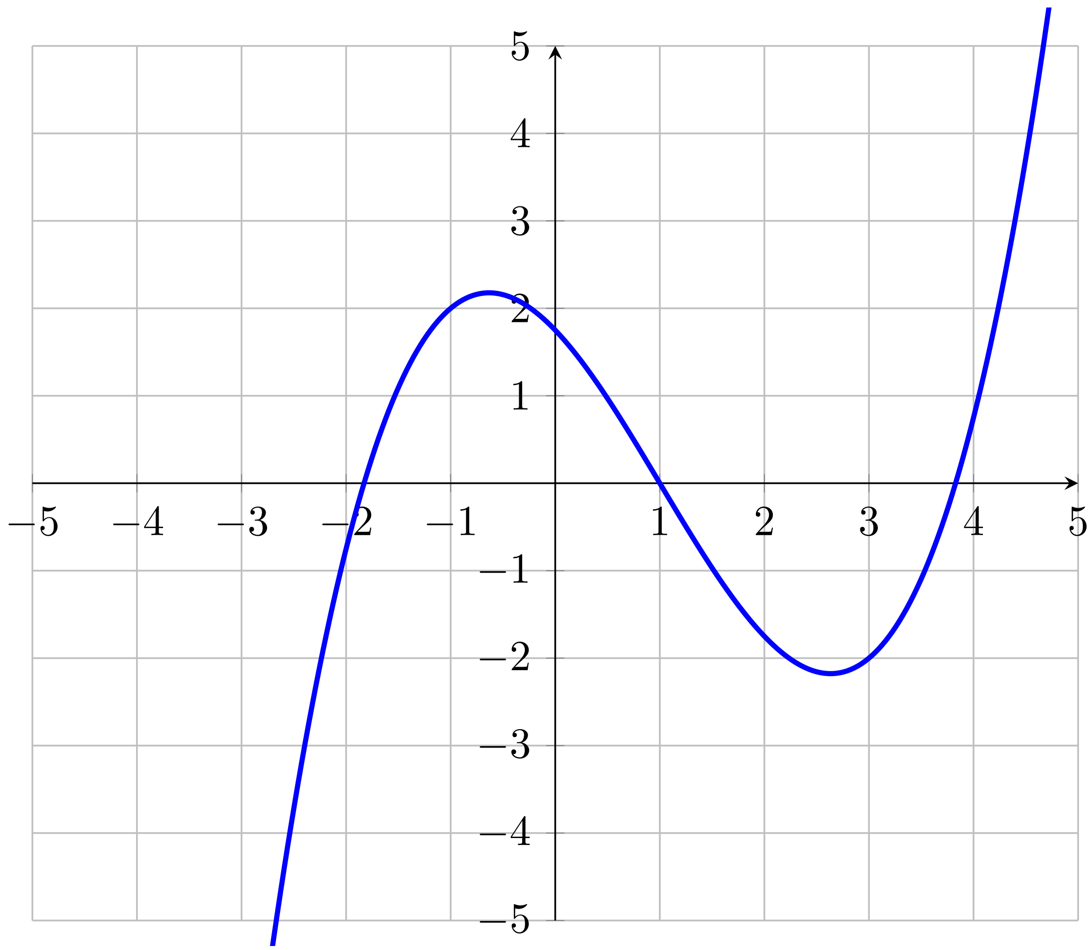
d. $f(t) = \sin(t)$
$\cos(t), (-\sin(t))$
ans

e. $f(t) = (\sqrt{t}+1)^2$
$1 + t^{-1/2}, -\frac{1}{2}t^{-3/2}$
ans

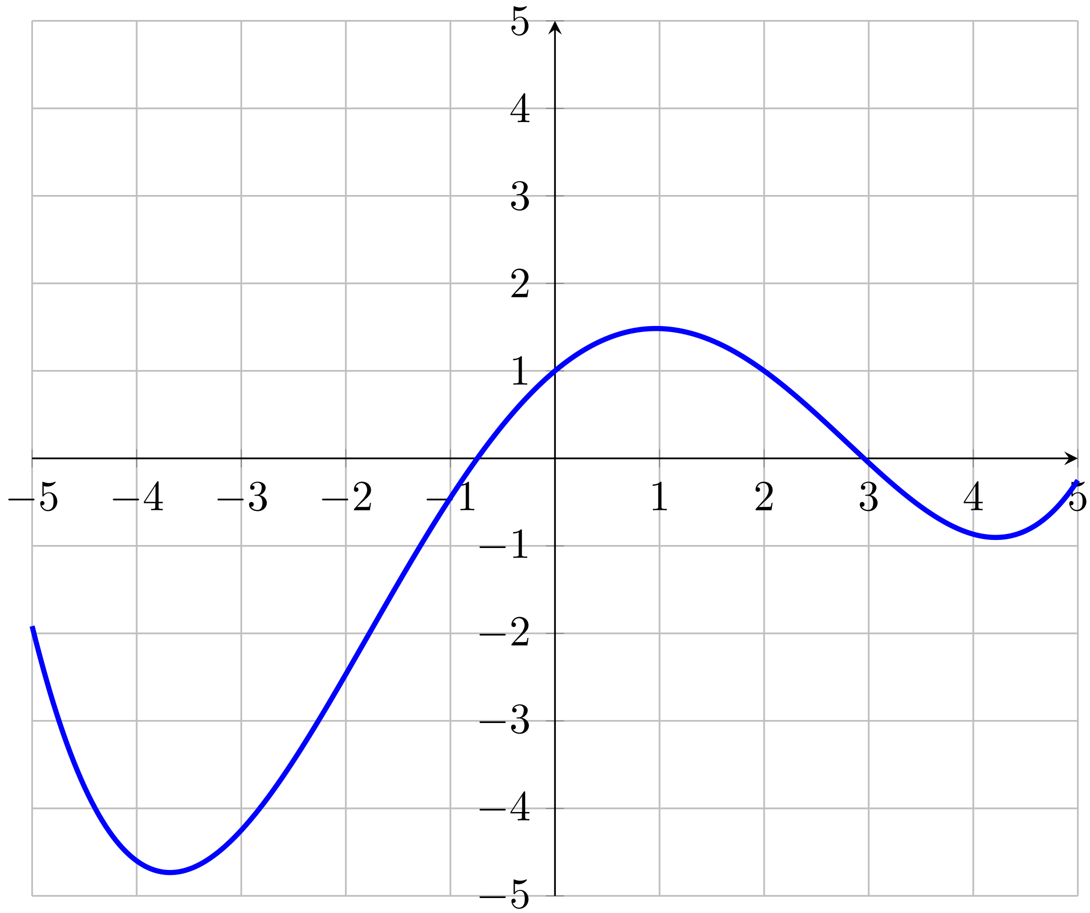
3. For each graph, circle any inflection points (if any). Label each region between the inflection points has having either a **positive** or **negative** second derivative.

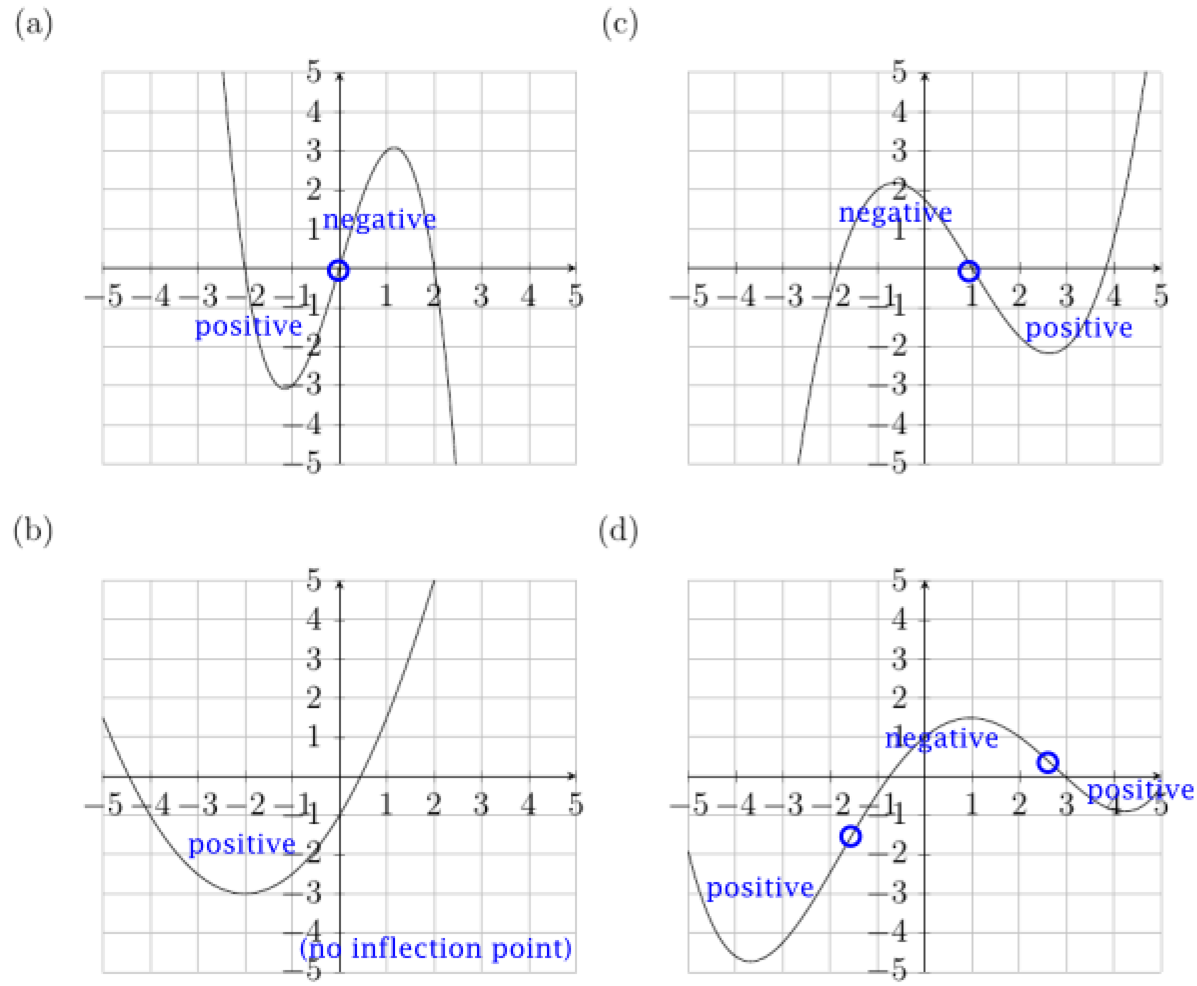
a. 

b. 

c. 

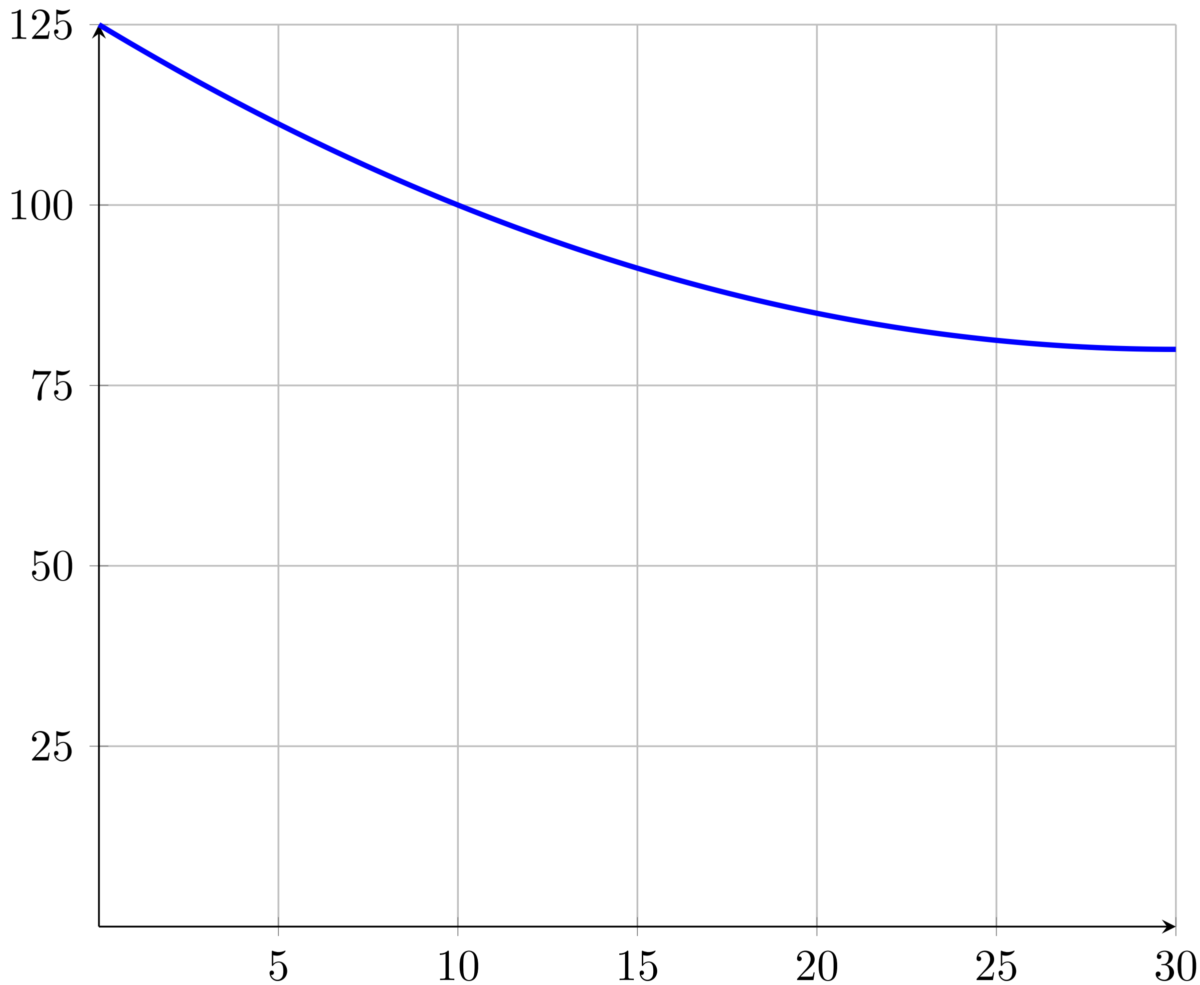
d.





ans

4. A river bank is eroding. Let $f(t)$ be the metric tonnes of soil and rock material on day t. Suppose $f(t) = 0.05t^2 - 3t + 125$ (this model is valid from $t = 0$ to $t = 30$) .



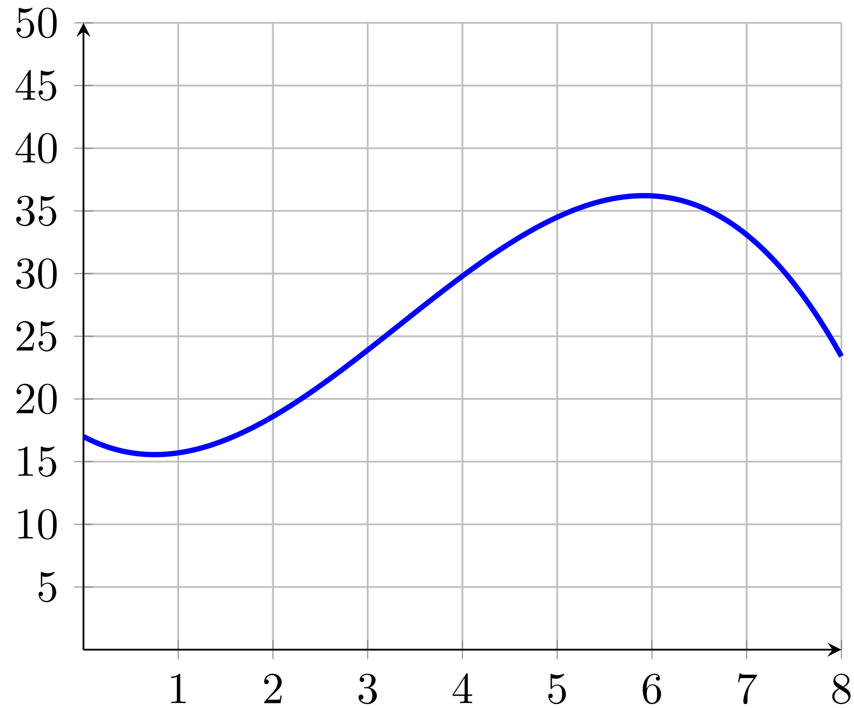
a. What is $f'(t)$ measuring? What are the correct units?
How quickly soil and rock are being lost due to erosion in metric tonnes per day.
ans

b. Sketch the derivative.

c. Find the derivative
$0.1x - 3$
ans

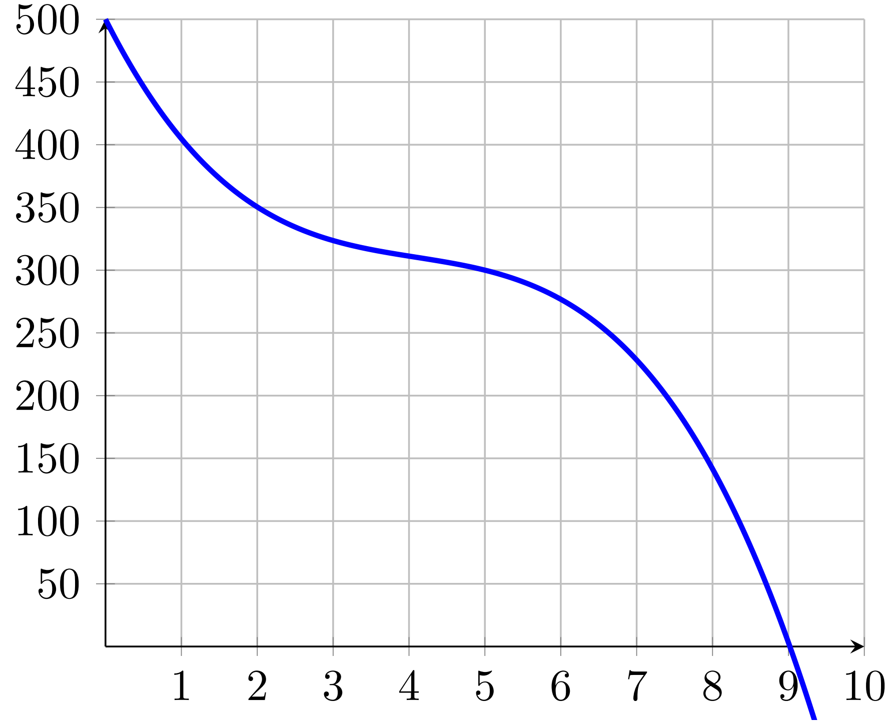
d. How quickly is material being lost on day $t = 5$? How quickly is material being lost on day $t = 25$?
2.5 metric tons per day on day 5, 0.5 metric tons per day on day 25.
ans

5. The stock price for Math Nerds, Inc, over the course of an 8 hour trading day ($t = 0$ to $t = 8$) is modeled by $p(t) = -0.3t^3 + 3t^2 - 4t + 17$ ($p(t)$ is measured in dollars).



a. What is $p'(t)$ measuring? What is $p''(t)$ measuring? State the correct units for each. $p'(t)$ is the rate of change of the stock price in dollars per hour. $p''(t)$ is how quickly the stock price rate is speeding up or slowing down, measured in dollars per hour^2
ans
b. Sketch the graph of $p'(t)$ and $p''(t)$).
c. Compute $p'(t)$ and $p''(t)$.
$p'(t) = -0.9t^2 + 6t - 4, p''(t) = -1.8t + 6.$
ans
d. How quickly is the stock gaining in price at $t = 4$? How quickly is it losing value at $t = 7$?
Gaining at 5.6 dollars per hour at $t = 4$, but losing at a rate of -6.1 dollars per hour at $t = 7$
ans
e. At $t = 4$, the stock price is clearly growing. But is growth speeding up or slowing down? How can you use the formula for $p(t)$, $p'(t)$, or $p''(t)$ to find out?
We can plug $t = 4$ into $p''(t)$, to get an answer of -1.2 dollars per hour per hour, which means growth is slowing down.
ans

6. A tank has $f(t)$ liters of water at time t measured in minutes, where $f(t) = -2.2t^3 + 27t^2 - 120t + 500$ (this model is valid $t = 0$ to $t = 9$).



a. Sketch the graphs of $f'(t)$ and $f''(t)$.
b. What is $f'(t)$ measuring? What is the meaning of $f''(t)$? Give the correct units.
Imagine water is leaking out of a hole. $f'(t)$ is indirectly measuring the size of that hole, since it is measuring how quickly water is being lost in liters per minute. $f''(t)$ would relate to how quickly the hole is opening or closing, thus measuring how fast the rate is changing in liters per minute per minute.
ans

CHAPTER 39

OPTIMIZATION

Sometimes we have a function and we just really want to know what its high point or low point is in terms of y-value. The high point is called a maximum, and the low point is called a minimum. For most functions, these points occur when the derivative is zero or undefined (we talked about why this is briefly in a previous section).

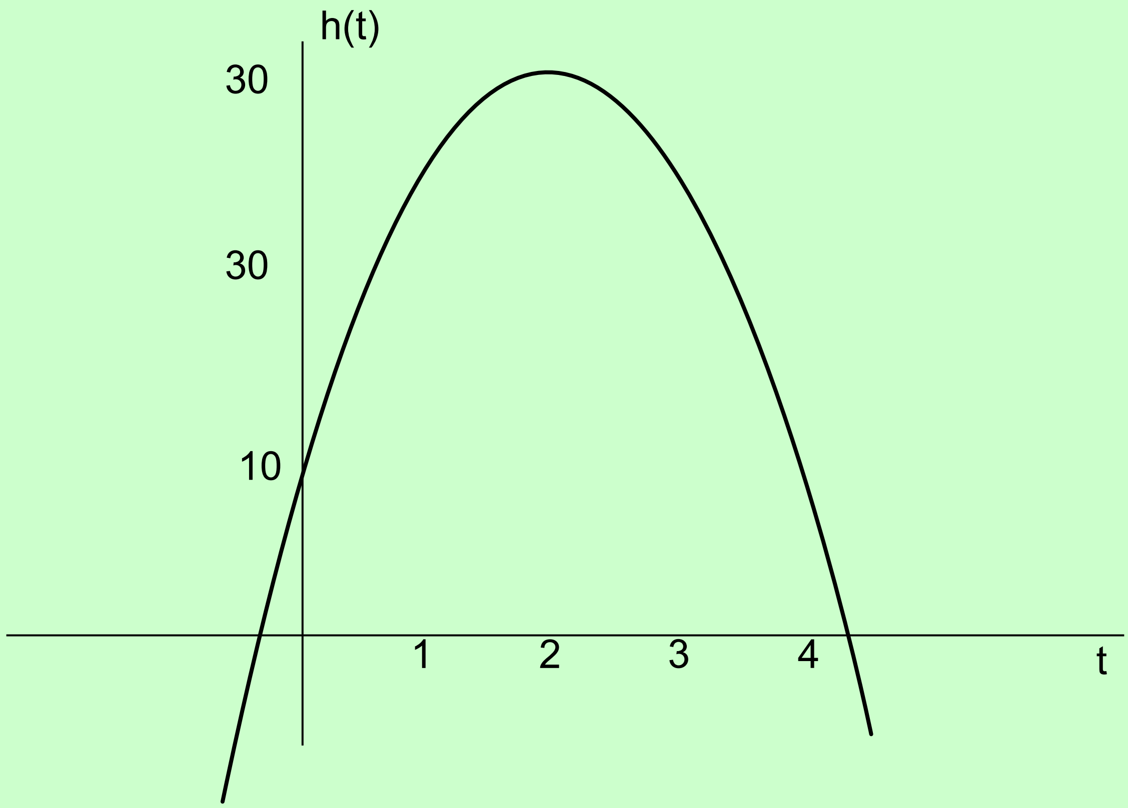
Example **Optimization**

Problem The height of a baseball follows the function $h(t) = -5t^2 + 20t + 10$, where h is measured in meters and t is measured in seconds.. What value of t maximizes the height?

We will follow the maxim "optimization happens when the derivative is zero". First we find the derivative using the power rule $h'(t) = -10t + 20$. Then we set this equal to zero, so we solve

$$
\begin{aligned}
-10t + 20 &= 0 \\
-10t &= -20 \\
t &= \boxed{2}
\end{aligned}
$$

Hence, the height is maximized when the time is equal to 2 seconds. At this point, the height of the ball is $h(2) = -5(2)^2 + 20(2) + 10 = -20 + 40 + 10 = 30$ meters. Here's a rough sketch based on what we know about this function:



Let's do anther optimization example:

Example **Optimization 2**

Problem The cost per item of producing Super Hero Action Figures, if x are produced, is given by

$$C(x) = \frac{500}{x} + 0.001x.$$

At what value x is the cost per item minimized? What is the cost at this value of x?

To solve this problem, we find $C'(x)$:

$$C'(x) = \frac{d}{dx}500x^{-1} + 0.001x = -500x^{-2} + 0.001$$

We then set this equal to zero and solve:

$$\begin{aligned}
-500x^{-2} + 0.001 &= 0 \\
-500x^{-2} &= -0.001 \\
x^{-2} &= 0.000002 \\
1 &= 0.000002x^2 \\
500,000 &= x^2 \\
\sqrt{500,000} &= x \approx \boxed{707.10}
\end{aligned}$$

So we see about 707 action figures is the best number to choose. The cost at this point (by plugging $x = 707$ into the original equation) is just $\boxed{\$1.41}$. Not too bad!

Chapter 40

HOMEWORK: OPTIMIZATION

1. Samantha has some whiskey at a party, and (being a science and math geek) estimates her blood alcohol content (BAC) follows the function:

$$BAC(t) = \frac{0.25t}{e^t},$$

where t is measured in hours after her first drink. Graph this function, and determine the following using a derivative:

   a. How quickly is her BAC increasing (or decreasing) 15 minutes after her first drink?
   $BAC'(t) = \frac{0.25-0.25t}{e^t}$, $BAC'(1/4) \approx 0.146$ (grams per dL per hour).
   ans
   b. How quickly is her BAC increasing (or decreasing) 1 hour after her first drink?
   0 change
   ans
   c. How quickly is her BAC increasing (or decreasing) 2 hours after her first drink?
   ≈ 0.034 grams per dL per hour
   ans

2. Graph each function over the given interval. Use calculus to determine the location of all global and local mins and maxes.

   a. $f(x) = -x^2 + 5x - 2$ on the interval $[0, 5]$.
   Local and Global Mins: $(0, -2)$, $(5, -2)$, Local and global max: $(2.5, 4.25)$
   ans
   b. $f(x) = x^2 - 6x + 10$ on the interval $[2, 6]$.
   Local min: $(2, 2)$, global and local min: $(3, 1)$, local and global max: $(6, 10)$
   ans
   c. $f(x) = x^3 - 6x^2 + 11x - 6$ on the interval $[0, 3]$.
   Local and global maximum at $(1.42, 0.38)$, Local min: $(2.58, -0.38)$, local and global minimum: $(0, -6)$, local maximum: $(3, -2)$
   ans
   d. $f(x) = x^3 - 5x^2 + 8x - 4$ on the interval $[0, 2.5]$.
   Local max at $x = 0.75$, local min at $x = 2$, global max at $x = 2.5$, global min at $x = 0$.
   ans
   e. $f(x) = x^4 - 16x$ on the interval $[0, 3]$.
   Local max at $x = 0$, global min at $x = \sqrt[3]{4}$, global max at $x = 3$

ans

f. $f(x) = \frac{x^2}{x+1}$ on the interval $[-3, 3]$.
Local min at $x = -3$, local max at $x = -2$, local min at $x = 0$, local max at $x = 3$
ans

3. Using a chemotherapy drug on a petri-dish of cancer cells, it is found that $P(x)$ percent more of the cancer cells are killed using x milligrams of drug per square centimeter than healthy cells, where x ranges from 0 to 4. It is thought

$$P(x) = x^3 - 8x^2 + 16x$$

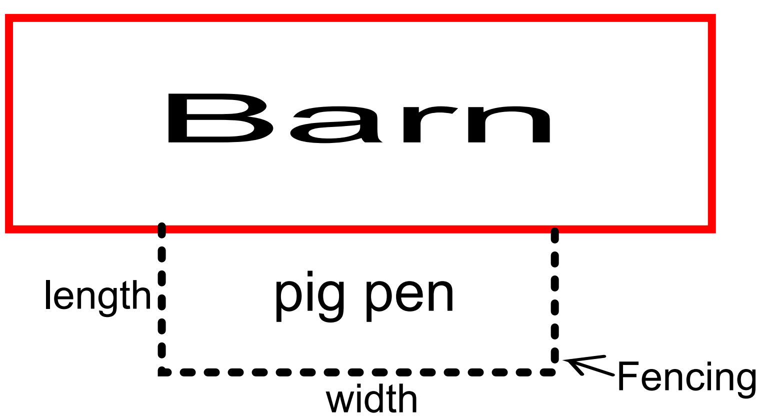
For what value of x is $P(x)$ maximized?
$x = 4/3$
ans

4. Bananas as we know them may be doomed! Suppose the fungus Tropical Race 4 mentioned in the article is killing off bananas on an island in Jamaica. The number of viable banana farms starts at 16000, with 800 being forced to close per year. But new banana farms are according to the function $20e^{0.3t}$ with new varieties immune to the fungus (t measured in years). So the total number of viable banana farms is

$$V(t) = 16000 - 800t + 20e^{0.3t}$$

on the interval $[0, 20]$. At what point is the number of banana farms minimized? What is the number of viable banana farms at this point?
This function is minimized at $t \approx 16.3$ with the number of banana farms at 5619
ans

5. The area of a rectangle is length times width. A farmer needs to build a pig pen against the side of the barn using 20 meters of fence. What is the maximum amount of area he can enclose?



6. Watch the KhanAcademy videos on maximizing the area of a box:
Optimizing Box Volume Graphically and
Optimizing Box Volume Analytically
7. An open-topped box is formed by removing the square corners of sidelength x off of a 40 in by 80 in piece of cardboard, and folding each side up. What value of x maximizes the volume of the box?
$x \approx 8.45$
ans

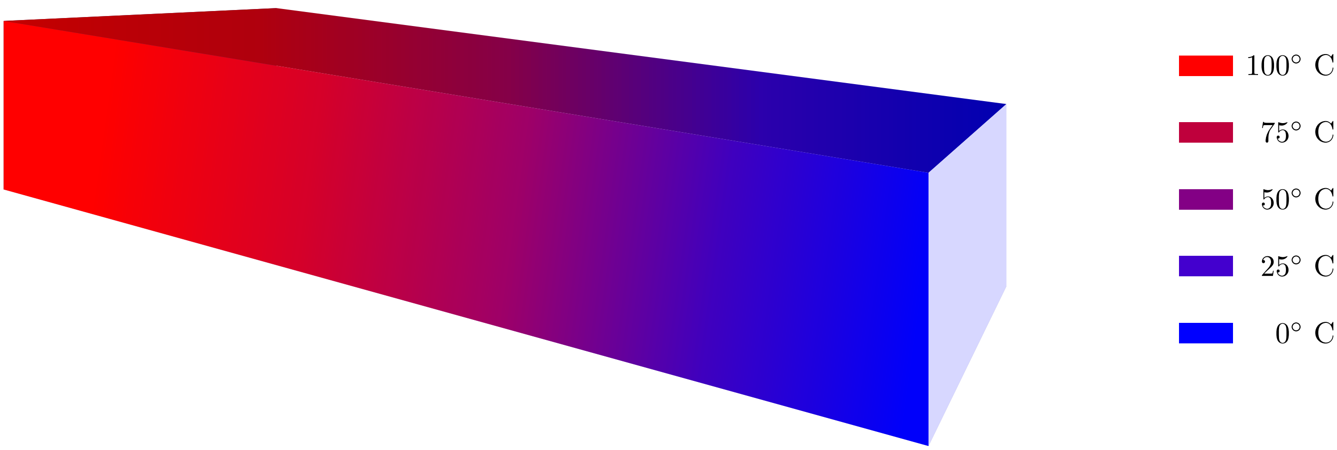
8. The height and radius of a cone together add to 5 inches. What value of the radius maximizes the volume? The volume is given by $V = \frac{1}{3}\pi r^2 h$.
   $r = 10/3 \approx 3.3$ in
   ans

CHAPTER 41

DERIVATIVES IN SPACE

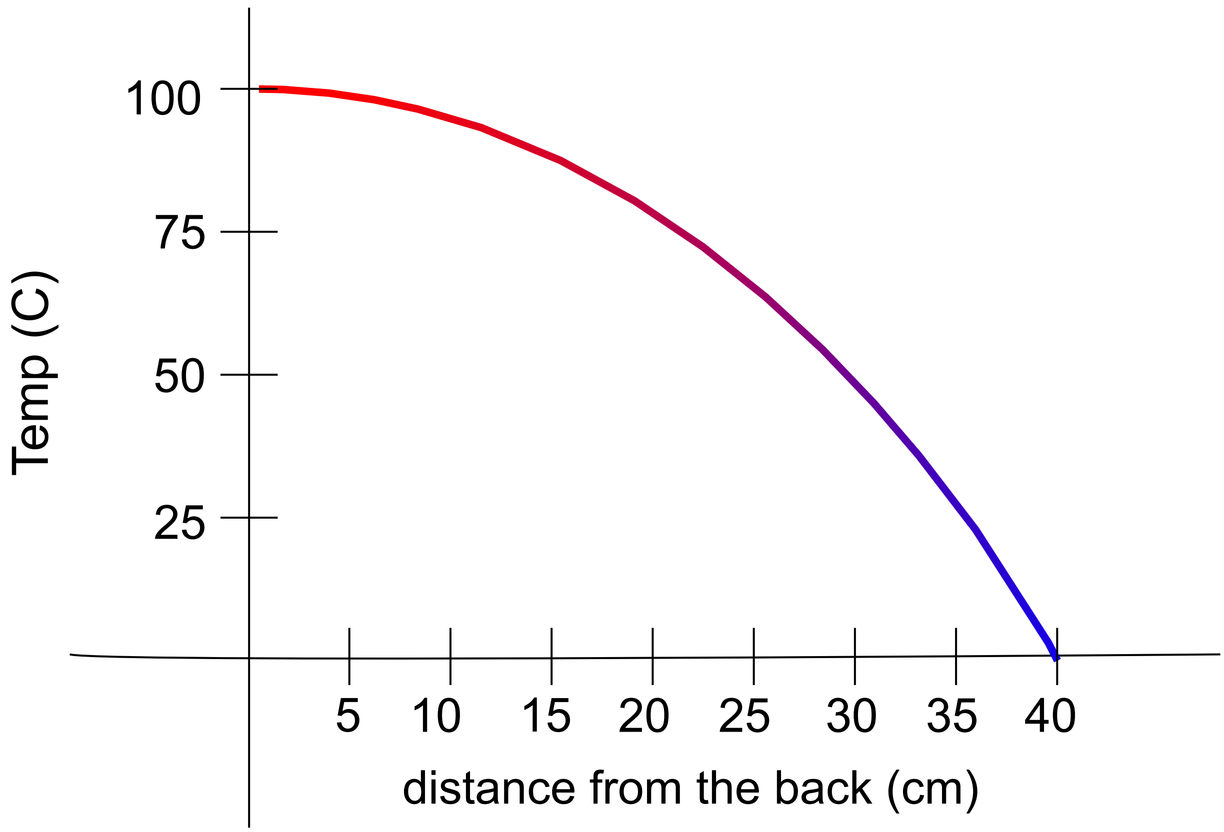
So far when we have thought about applying derivatives, we've always thought in terms of time. So we've focused on rates like meters per second, degrees Celsius per second, liters per minute, and so on, everything per unit time. However, we don't always get derivatives with respect to time.

For example, consider this box:



And let's consider the temperature of the box at every point. We can show this using red to mean hot and blue to mean cold:

Note that the temperature changes, **but not in time**. Instead, the temperature changes as you move down the box — that is, the temperature changes **in space**. We can show the same thing in the graph:



And just as before, the slope of this graph at any point is the derivative. What are the units? Well, they would be $\frac{\text{C}^\circ}{\text{cm}}$.

Let's continue with this example.

Example **Temp in a box**

Problem Suppose the equation $T(x) = -\frac{1}{16}x^2 + 100$ described the temperature, in Celsius, of the box from the previous diagrams at a distance of x centimeters from the back.

1. What is the derivative of the temperature halfway down the box?
2. What is the derivative of the temperature three-quarters of the way down the box?
3. Let's go back to thinking about time. According to the **heat equation**, a point in space will tend to get hotter if the second derivative in space is positive, and colder if the second derivative in space is negative. Based on this statement, is this box going to get hotter or colder? What does that mean regarding its derivative in time?

1. We can see from the previous graph that the box is 40 centimeters long, so half way would be $x = 20$ cm. We know $T'(x) = -\frac{1}{16}(2x) = -\frac{1}{8}x$. So if we just plug in $x = 20$, we get
$T'(20) = -\frac{1}{8}(20) = -2.5$. Again, the units are $\frac{\text{C}^\circ}{\text{cm}}$. So that means for every cm you travel, the box gets 2.5° C colder.
2. Now we just plug in $x = 30$ into the derivative we already found $T'(x) = -\frac{1}{8}x$, and

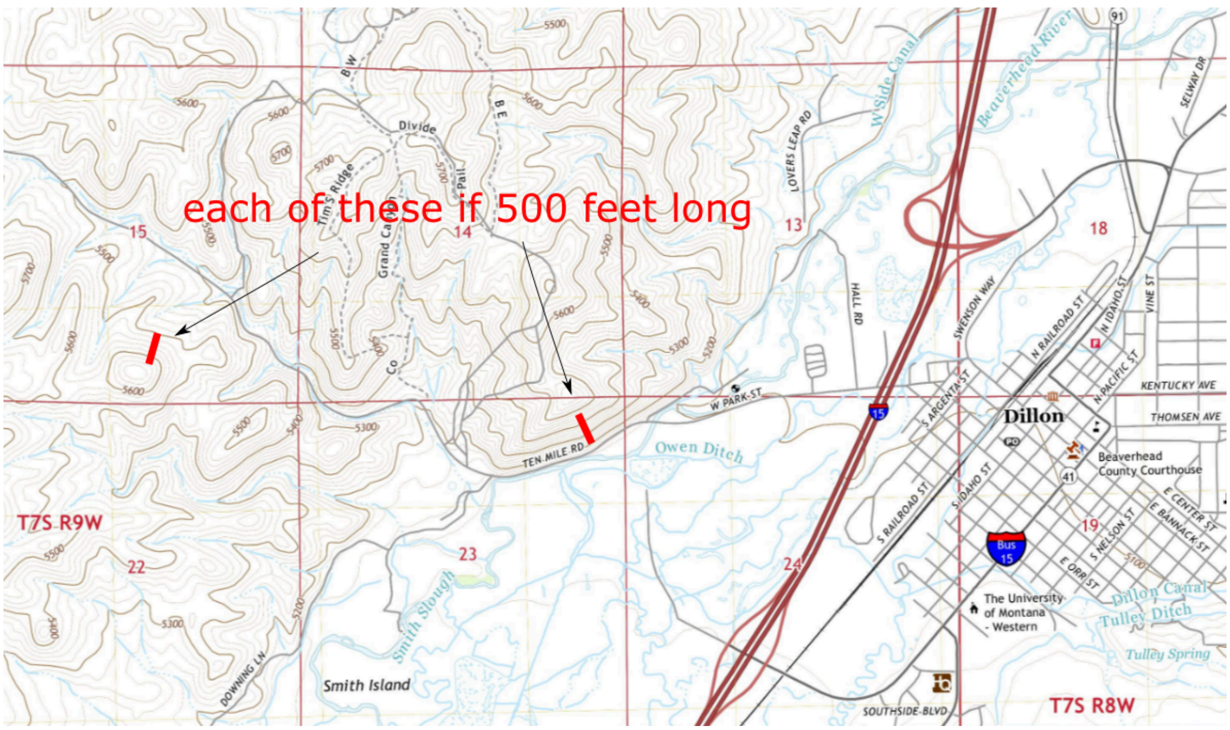
we see $T'(30) = -\frac{1}{8}(30) = -3.75$. This means, at this point in the box, the box is getting colder faster. You can see this in the graph as well.

3. To answer this question, we need the second derivative $T''(x)$. This is just the derivative of $T'(x) = -\frac{1}{8}x$, and so we just drop the x and get $T''(x) = -\frac{1}{8}$. Since the second derivative with respect to space is negative, the heat equation says the first derivative with respect to time is also negative. So if we just leave this box alone, it would tend to get colder. That is, the $T'(t)$ is negative as well.

Chapter 42

HOMEWORK: DERIVATIVES IN SPACE

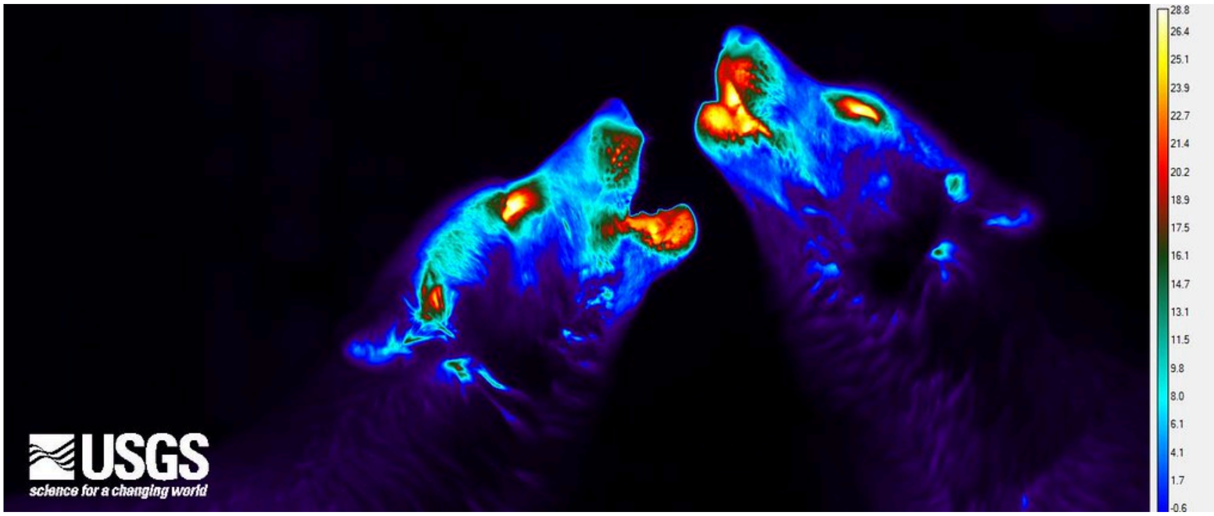
1. We've talked a lot about how derivatives measure slope, but literal slope on a topographic map is a good example of derivatives in space. Consider this picture:



src: usgs.gov's topoview

The thinnest contours on the map represent a height gain of 20 feet, and the slightly thicker contours are a height gain of 100 feet. Each thick red line has a length of roughly 500 feet. What is the slope of the mountain at the thick red lines?

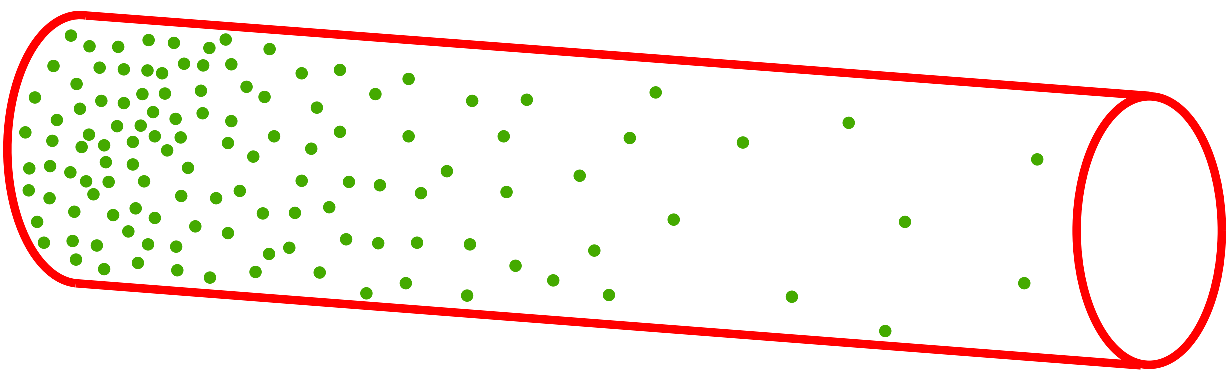
2. Consider the following thermal image of wolves howling:



src: usgs.gov

a. Why is the eyes and mouth much hotter than other parts of the wolves, like the neck?
b. According to the heat equation, things get colder or warmer depending on the second derivative with respect to space. Assuming these wolves are a constant temperature, that means the second derivative of temperature with respect to space is zero. What does it mean for the first derivative if the second derivative is zero? What does that mean for the temperature of the wolves at different places on its body where the fur is thinner or thicker?

3. Consider a cylinder filled with particles of some important nutrient (say oxygen). Suppose the concentration isn't constant, but instead looks something like this:



The oxygen particles will tend to move from areas of high concentration to areas of low concentration, a process called diffusion. Diffusion is extremely important in biology, since that is how many cells get their nutrients.

a. Based on the diagram of the cylinder, draw a rough sketch of the concentration graph. On the y-axis should be concentration C (molecules per cm^3), and on the x-axis should be distance x from the back of the cylinder measured in cm.
b. Based on your graph, estimate the slope at two points of your choosing. The notation for your answers would be $\frac{dC}{dx}$.

c. The rate at which the particles move in diffusion is called flux. It follows Fick's law:

$$\text{Flux} = -D \cdot \frac{dC}{dx}$$

Here, flux is how fast the particles are moving, D is the diffusion coefficient (a constant), and $\frac{dC}{dx}$ is what we found in the previous part. Why is there a negative in the equation? Explain it in words, and talk about your graph and your answers for $\frac{dC}{dx}$ from the previous part.

PART V

DIFFERENTIAL EQUATIONS

CHAPTER 43

RECURRANCE RELATIONS

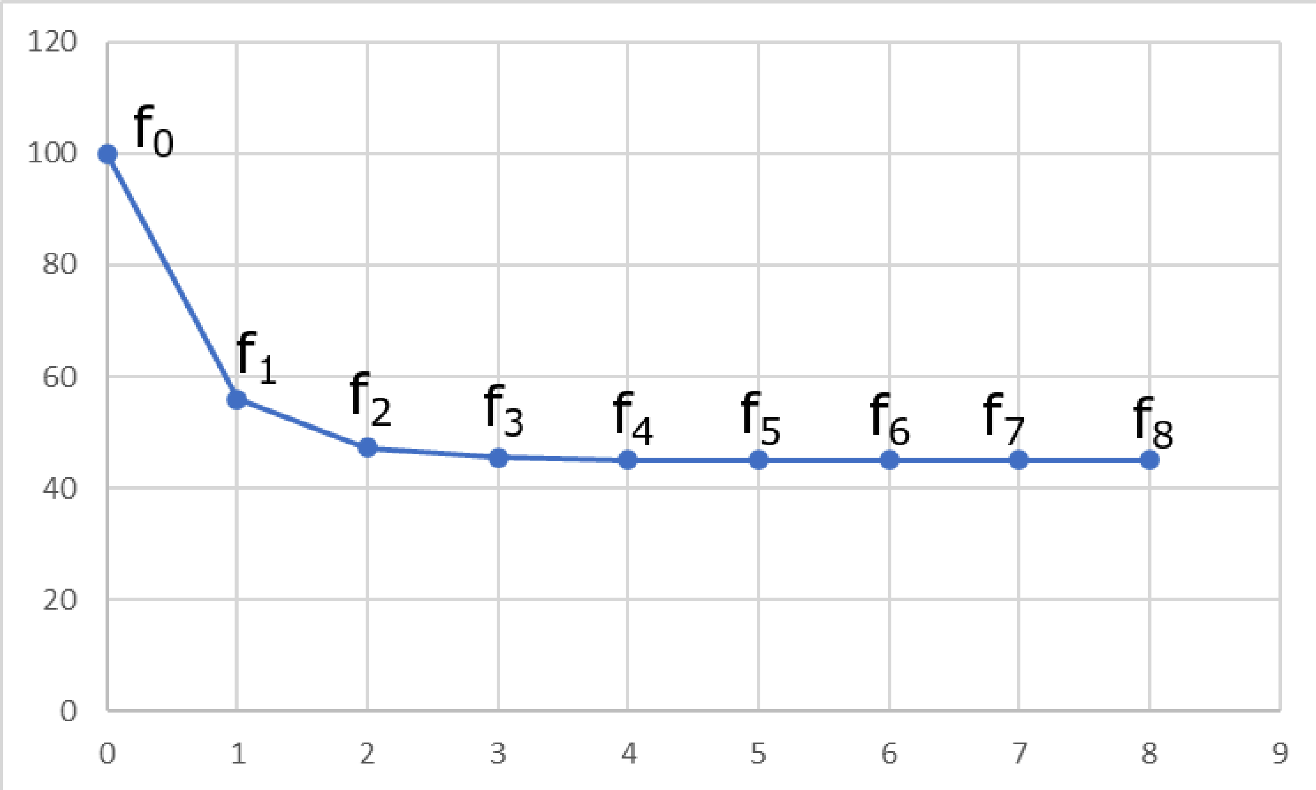
As we'll see in the next section, a differential equation looks like this: $\frac{dP}{dt} = 0.03 \cdot P$. What I want to first talk about though are *recurrence relations*. Let me introduce these with a magic trick.

Example **Magic Trick**
Pick a number between 1 and 100, and I'm going to guess it. But not before we mix it up a bit.

1. Take your number and divide by five, and round to the nearest whole number.
2. Then add 36 to the result.
3. Repeat steps (1) and (2) twice more, for a total of three iterations.

Done? I bet you ended with the number 45. Are you amazed?

This trick is based off the recurrence relation $f_{t+1} = \frac{1}{5} f_t + 36$. Think of f_t as the previous value, and f_{t+1} as the new value. How do you get from one to another? Well, ignoring the rounding, you divide by 5 and add 36, and that's exactly what $f_{t+1} = \frac{1}{5} f_t + 36$ is telling you to do. In order to use such a equation, we need an *initial value* or f_0. In the trick, this was the original number you picked. Let's create a graph with the initial value $f_0 = 100$:



As you can see, this recurrence relation quickly converges to $f_t = 45$ by the time $t = 3$. That's why the trick works! If we started somewhere else, the graph looks much the same and it converges to 45 anyway.

However, recurrence relations are useful for more than just magic tricks.

Example **Logging**

Problem Let h_t be the biomass of a forest in year t. Suppose it expands by 1% each year, but also loses 2000 metric tonnes to logging. What might be a recurrence relation that explains this situation?

Well, since this is a recurrence relation, we want to relate the quantity under consideration, h_t to its value the next year, which is h_{t+1}. So it will look something like

$$h_{t+1} = 1.5h_t + 16$$

but those aren't the right values yet — just want to have some idea of where this is going.

The first thing we need to encode is the expansion by 1%. We can take 1%, or 0.01 and multiply by h_t like so: $0.01h_t$. But the old forest is still there (except for the logging, which we'll worry about in a second), so let's add h_t as well: $0.01h_t + h_t$. If we factor out h_t, we get

$$\begin{aligned} 0.01h_t + h_t &= h_t(0.01 + 1) \\ &= h_t(1.01) \\ &= 1.01h_t \end{aligned}$$

This is the growth by 1%. What about that logging? Well, that's not a percent change, so we'll just subtract the 2000 to represent the loss of biomass. So our final recurrence relation is–>

$$h_{t+1} = 1.01h_t - 2000$$

Problem What happens to the forest in the long run according to your recurrence relation?

Well, let's play with it a bit and see what happens. But before we can do that, we need an initial value h_0. Let's guess something. Since we are losing 2000 a year, we'll need a much bigger number than 2000. Let's just guess that h_0 is $50{,}000$ metric tonnes.
Now we can compute several h_t values:

$$\begin{aligned} h_0 &= 50000 \\ h_1 &= 1.01(50000) - 2000 = 48500 \\ h_2 &= 1.01(48500) - 2000 = 46985 \\ h_3 &= 1.01(46985) - 2000 \approx 45500 \end{aligned}$$

We can see that the biomass is going down — not a good sign for the forest. We can speed these calculations up quite a bit in excel. If you do that, you can see that the forest will be totally gone in by h_{29}, in less than thirty years. However, that's not a full answer, since it may depend on how much biomass we start with. Suppose it's a larger forest with $h_0 = 300{,}000$. Then we see

$$\begin{aligned} h_0 &= 300000 \\ h_1 &= 1.01(300000) - 2000 = 301000 \\ h_2 &= 1.01(301000) - 2000 = 302010 \\ h_3 &= 1.01(302010) - 2000 \approx 303000 \end{aligned}$$

And the forest just grows from there.

Let's see an even more complicated example.

Example **Life Cycle of Cutthroat Trout**

Problem Recurrence relations can model the life of plant and animals species as they move from one stage of life to the next. For example, let f_t^0, f_t^1, f_t^2, f_t^3, f_t^4, f_t^{5+} be the amount of cutthroat trout (oncorhynchus clarkii) in southwest Montana of age 0, 1, 2, 3, 4, and 5 or more respectively. Here, t is a measured in years starting from some $t = 0$. Then according to https://compadre-db.org/Species/47501, these quantities follow the recurrence relation

$$
\begin{aligned}
f^0_{t+1} &= 5.23 f^3_t + 18.0 f^4_t + 24.55 f^{5+}_t \\
f^1_{t+1} &= 0.277 f^0_t \\
f^2_{t+1} &= 0.3405 f^1_t \\
f^3_{t+1} &= 0.4675 f^2_t \\
f^4_{t+1} &= 0.4675 f^3_t \\
f^{5+}_{t+1} &= 0.4675 f^4_t + 0.4675 f^{5+}_t
\end{aligned}
$$

Explain what each number in these recurrence relations mean.

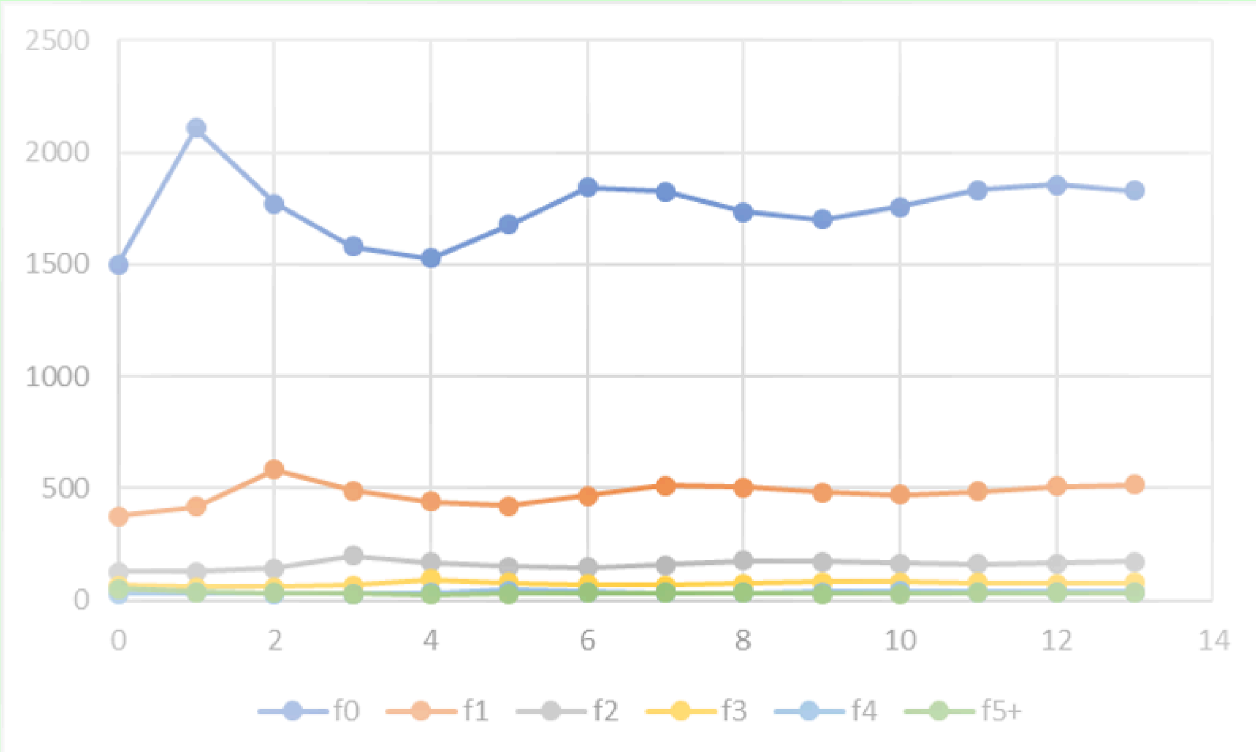
My goodness, that's a complicated mess of symbols. But with a little patience, we can figure it out I think.

Let's start with the line $f^1_{t+1} = 0.277 f^0_t$. We know from the problem statement that f^0_t are the trout of age 0 at year t. The quantity f^1_{t+1} is the amount of one year old trout at year $t+1$. This equation is relating the number of 0 year olds with the number of 1 year olds a year later. What it is saying is 0.277 times the number of zero year olds gives you the number of one year olds a year later. In other words, this equation is giving a 27.7% survival rate from age zero to age one. From here, we can now easily decode several other equations. $f^2_{t+1} = 0.3405 f^1_t$ gives a 34.05% survival rate from age one to age two. The similar we find 46.75% survival rate from age two to age three, and the same rate from age three to age four. The equation $f^{5+}_{t+1} = 0.4675 f^4_t + 0.4675 f^{5+}_t$ is a bit more complicated, since trout of age 5+ come from the age 4 trout, but also the age 5+ stay within that category, so there are two ways to get there. Both of these involve a 46.75% survival rate.

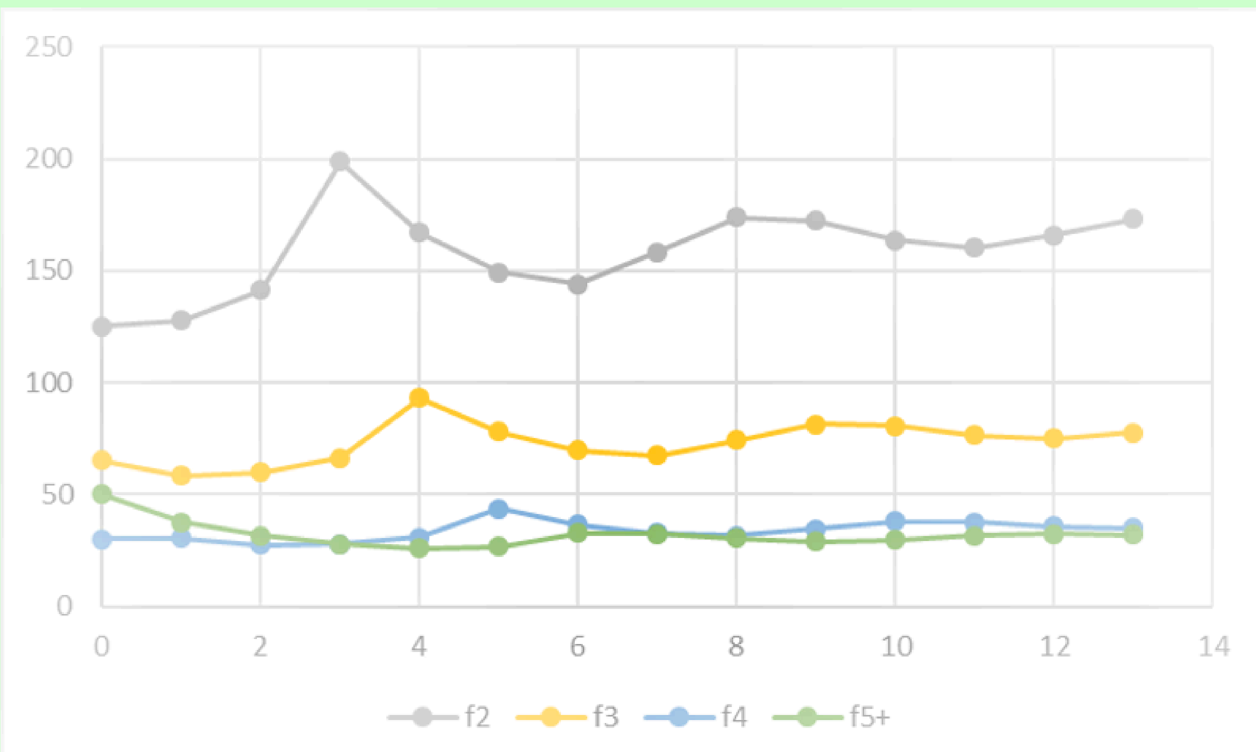
Notice these survival rates are pretty low by human standards. However, there is some good news for the species: look at the first equation $f^0_{t+1} = 5.23 f^3_t + 18.0 f^4_t + 24.55 f^{5+}_t$. What do you make of this? That's right — these are the new baby trout! As you can see, there are a lot of new babies that help balance the low survival rates we noticed before. In particular, each three year old produces roughly 5 new offspring, each four year old produces on average 18 new offspring, and an average 5+ year old produces almost 25 offspring.

Problem Starting with $f^0_0 = 1500$, $f^1_0 = 375$, $f^2_0 = 125$, $f^3_0 = 65$, $f^4_0 = 30$, and $f^{5+}_0 = 50$, create a graph that shows how the population of the cutthroat trout changes over time. What does the graph show?

I created this graph in Excel (see the file "cutthroat-life-cycle.xlsx" for the formulas and data):



And here are just the last four stages to get a better look at these ones:



As we can see, the population seems to be fairly stable. One thing that stands out to me is how many age zero and age one fish there are compared to other groups.

Chapter 44

HOMEWORK: RECURRENCE RELATIONS

1. For the logging example from previous section, for initial forest size h_0 is the biomass stable? That is, the biomass of the forest is not changing from year to year (this is known as a *fixed point*).
2. Find a species of your choosing on the website https://compadre-db.com/ExploreDatabase, like we did for the cutthroat trout in the previous section. Use a spreadsheet program to see what happens to the population over time. What do you notice?

Chapter 45

Introduction to Differential Equations

Sometimes we don't quite know what kind of function we are dealing with exactly, but we know some basic things about its derivative. For example, consider how many people live in a town or city. We'll call this $P(t)$. One general principle that often holds true for population is that the rate it is growing is proportional to the size of the population. That is, *the bigger the city, the faster the growth*. This is just saying in terms of raw numbers, a city like Hong Kong has the ability to add people much faster than Dillon, Montana, since Hong Kong is a much bigger city. Now, this doesn't have to hold: some big cities actually shrink, while some small towns explode in population overnight. But on average, this is true. It follows the equation

$$\frac{dP}{dt} = 0.03 \cdot P$$

The left hand side represents how fast P is growing, and the right hand side represents some fraction of P. The value 0.03 is called the growth rate. This is an example of a differential equation. A *differential equation* is an equation relating a unknown function and its derivatives. Another way to write the same differential equation is to use Newton's notation.

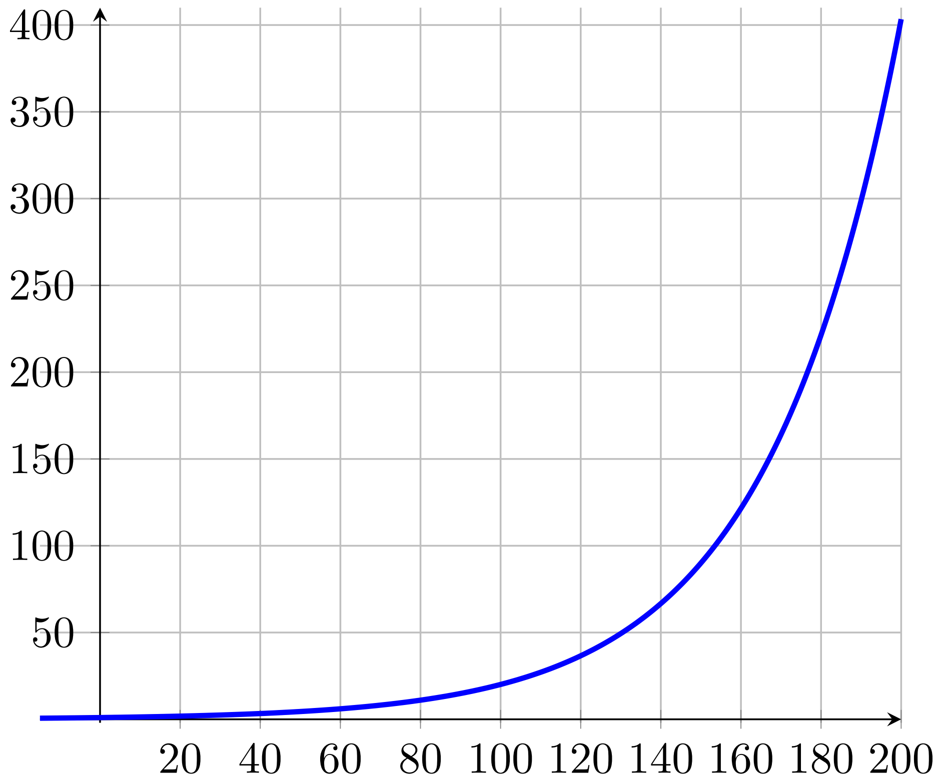
$$P'(t) = 0.03 \cdot P(t)$$

Again, this is just saying how fast $P(t)$ is growing is equal to some constant times the size of $P(t)$.

We'll learn how to "solve" an differential equation later on. But for now, note that $P(t) = e^{0.03t}$ solves this differential equation. Why? We'll, note that $P'(t) = 0.03e^{0.03t}$ by the chain rule. Hence, we can verify that $P(t) = e^{0.03t}$ solves this differential equation by using substitution.

$$\begin{aligned} P'(t) &= 0.03 \cdot P(t) \\ (0.03e^{0.03t}) &= 0.03 \cdot (e^{0.03t}) \\ 0.03e^{0.03t} &= 0.03e^{0.03t} \end{aligned}$$

Since we get the same thing on both sides via substitution, we know that the differential equation is verified! What does this mean? Well, this says that population follows the exponential function $e^{0.03t}$. This means populations eventually start to grow extremely fast.



While exponential functions grow bigger and bigger forever, in practice, population growth will eventually slow or even stop due to geographic or other constraints. *No exponential function lasts forever in real life!*

Let's see a couple other examples of creating differential equations.

Example **Creating Differential Equations**

- Problem Let $E(t)$ be the earnings of a large company, measured in millions of dollars. This company's growth in earnings is 0.07 times their current earnings. What is a differential equation that models this situation?

We see that growth in earnings is the same thing as derivative. Hence $E'(t)$ is the growth in earnings, and this is equal to 0.07 times their current earnings, so $\boxed{E'(t) = 0.07 \cdot E(t)}$.

- Problem The growth of a function is 2.5 less than 0.3 times the value of the function. What is the differential equation now?

We see that this can be translated as $\boxed{f'(t) = 0.3f(t) - 2.5}$.

Here are some examples of verifying a given solution is correct.

Example

Problem Verify that $f(x) = e^x + x + 1$ solves the differential equation:

$$f'(x) = f(x) - x$$

Here, we see $f(x) = e^x + x + 1$ is given, and we can compute this by taking the derivative of each piece.

$$\begin{aligned} f'(x) &= \frac{d}{dx}e^x + x + 1 \\ &= e^x + 1 \end{aligned}$$

We can then verify the differential equation using subsitution.

$$\begin{aligned} f'(x) &= f(x) - x \\ e^x + 1 &= (e^x + x + 1) - x \\ e^x + 1 &= e^x + 1 \end{aligned}$$

Example

Problem Verify that $q(x) = \sqrt{2x}$ solves the differential equation:

$$q'(x) = \tfrac{1}{q(x)}$$

Here, we see $q(x) = \sqrt{2x}$ is given, and we can compute using the chain rule

$$\begin{aligned} q'(x) &= \frac{d}{dx}\sqrt{2x} \\ &= \frac{d}{dx}(2x)^{1/2} \\ &= \frac{1}{2}(2x)^{-1/2} \cdot 2 \\ &= (2x)^{-1/2} \\ &= \frac{1}{\sqrt{2x}} \end{aligned}$$

We can then verify the differential equation using substitution.

$$
\begin{aligned}
q'(x) &= \frac{1}{q(x)} \\
\left(\frac{1}{\sqrt{2x}}\right) &= \frac{1}{(\sqrt{2x})} \\
\frac{1}{\sqrt{2x}} &= \frac{1}{\sqrt{2x}}
\end{aligned}
$$

CHAPTER 46

HOMEWORK: INTRODUCTION TO DIFFERENTIAL EQUATIONS

1. Describe as best you can at this point in your own words what a differential equation is.
2. Following earnings example from the previous chapter, if the number of employees in a company is growing at a rate of 0.05 times the number of employees, what is a differential equation that describes this situation?
   $E'(t) = 0.05E(t)$.
   ans
3. Verify the function $f(x) = e^x - x - 1$ solves the differential equation:
   $$f'(x) = f(x) + x$$
   We see
   $$\begin{aligned} f'(x) &= f(x) + x \\ e^x - 1 &= (e^x - x - 1) + x \\ e^x - 1 &= e^x - 1 \end{aligned}$$
   as desired.
   ans
4. Verify the function $f(x) = 2\sqrt{x}$ satisfies the differential equation:
   $$f'(x) = \tfrac{2}{f(x)}.$$
   We see
   $$\begin{aligned} f'(x) &= \frac{2}{f(x)} \\ x^{-1/2} &= \frac{2}{2\sqrt{x}} \\ \frac{1}{\sqrt{x}} &= \frac{1}{\sqrt{x}} \end{aligned}$$
   as desired.
   ans

5. For each differential equation, find $f'(t)$ for the given value of t, or state there is not enough information.

    a. Suppose $f'(t) = 3f(t) + 5$ and $f(3) = -1$. Find $f'(3)$.
    2
    ans
    b. Suppose $f'(t) = t + f(t)$, and $f(7) = 1$. Find $f'(7)$.
    8
    ans
    c. Suppose $f'(t) = \frac{1}{\sqrt{f(t)}}$ and $f(0) = 9$. Find $f'(0)$.
    $\frac{1}{3}$
    ans
    d. Suppose $f'(t) = e^{-f(t)}$ and $f(0) = 1$. Find $f'(1)$.
    Not enough information.
    ans

6. For each relationship between the value of a function and its derivative, write down a differential equation. For example, if I said "a function is growing at a rate equal to seven times the value of the function" you'd write down $f'(t) = 7f(t)$.

    a. A function is growing at a rate equal to twice the function value.
    $f'(t) = 2f(t)$
    ans
    b. A function is growing at a rate equal to the square root of the function value.
    $f'(t) = \sqrt{f(t)}$
    ans
    c. A function is growing at a rate equal to t times the function value.
    $f'(t) = tf(t)$
    ans
    d. A function is accelerating at a rate equal to the sum of the function value and how quickly the function is growing.
    $f''(t) = f'(t) + f(t)$.
    ans

7. Verify that the given solution to each differential equation is correct.

    a. Differential equation $f'(t) = f(t) + 3$, solution $f(t) = 3e^t - 3$.

    $$\begin{aligned} f'(t) &= f(t) + 3 \\ \frac{d}{dt}(3e^t - 3) &= (3e^t - 3) + 3 \\ 3e^t &= 3e^t \end{aligned}$$

    ans
    b. Differential equation $f'(t) = 4\sqrt{f(t)}$, solution $f(t) = 4t^2$.

$$\begin{aligned} f'(t) &= 4\sqrt{f(t)} \\ \frac{d}{dt}(4t^2) &= 4\sqrt{4t^2} \\ 8t &= 4(2t) \\ 8t &= 8t. \end{aligned}$$

ans

c. Differential equation $f'(t) = (f(t))^2$, solution $f(t) = -t^{-1}$.

$$\begin{aligned} f'(t) &= (f(t))^2 \\ \frac{d}{dt}(-t^{-1}) &= (-t^{-1})^2 \\ t^{-2} &= t^{-2} \end{aligned}$$

ans

d. Differential equation $f'(t) = e^{-f(t)}$, solution $f(t) = \ln(t)$.

$$\begin{aligned} f'(t) &= e^{-f(t)} \\ \frac{d}{dt}\ln(t) &= e^{-\ln(t)} \\ \frac{1}{t} &= \frac{1}{e^{\ln(t)}} \\ \frac{1}{t} &= \frac{1}{t} \end{aligned}$$

ans

CHAPTER 47

UNDERSTANDING DIFFERENTIAL EQUATIONS

We will now outline a process one can follow to make sense of a difficult or complicated differential equation:

1. Understand what each variable is measuring with correct **units**.
2. Write down **what** relationship the differential equation is describing in common, no-nonsense language.
3. Explain **why** those relationships seem to make sense.

Let's do an example with the population equation.

Example **Simple Population Growth**

Problem A population of snakes is declining according to the model $S'(t) = -0.05S(t)$, where $S(t)$ is number of snakes, and t is measured in years. Follow the steps above to explain this differential equation.

1. *Understand what each variable is measuring with correct* ***units****.*

   As given in the problem, $S(t)$ is number of snakes at time t, t is measured in years. Since $S'(t)$ is a rate of change of the snakes, it is snakes lost per year.
2. *Write down* ***what*** *relationship the differential equation is describing in common, no-nonsense language.*

   The population of snakes is decreasing, because of the negative sign, in proportion to the number of snakes. Hence, a good answer here is "The more snakes you have, the more snakes you lose."
3. *Explain* ***why*** *those relationships seem to make sense.*

   Of course we don't know why the snakes are dying off, but it makes sense that the more snakes you have the more you lose, since there are more snakes with the potential to die.

The next example involves a system of differential equations which makes it a bit more complicated. A system of differential equations is just like a system of equations: you now have perhaps several unknown functions, and you want to find all the unknown functions so that all the equations are true at the same time.

Example **Wolves and Deer**

Problem Let $W(t)$ be the population of wolves and $D(t)$ be the population of deer, with t measured in years. They satisfy the differential equation

$$D'(t) = -0.1(W(t) - 30)$$
$$W'(t) = 0.1(D(t) - 300).$$

Explain this differential equation.

1. *Understand what each variable is measuring with correct* ***units****.*

   As given in the problem, $D(t)$ is the number of deer at a given time, $W(t)$ the number of wolves, and t is measured in years. $D'(t)$ is how fast the deer population is changing in deer per year, and $W'(t)$ the same thing for wolves measured in wolves per year.
2. *Write down* ***what*** *relationship the differential equation is describing in common, no-nonsense language.*

   We see that a wolf population larger than 30 makes the deer population go down. A deer population above 300 makes the wolf population go up.
3. *Explain* ***why*** *those relationships seem to make sense.*

   These relationships make sense since if we have a lot of wolves, they hunt the deer and the population of deer goes down. If we have a lot of deer, there is a lot of food for the wolves, so the wolf population goes up.

Let's look at a rocket equation.

Example **Rocket Equation**

Problem A rocket that is blasting off has height $h(t)$ in meters and has mass of fuel $m(t)$ in kg (here, t is measured in seconds). These quantities follow the differential equations

$$h''(t) = \frac{-10000m'(t)}{50 + m(t)} - 9.8$$
$$m'(t) = -0.1$$

Use the steps to explain this differential equation.

1. *Understand what each variable is measuring with correct* ***units****.*

   As given in the problem, $h(t)$ is the height measured in meters, $m(t)$ is the mass of fuel in kg, and t is time measured in seconds. We also have $h'(t)$ would be the velocity in the upward direction in m/s, and $h''(t)$ is acceleration upward of the rocket in m/s^2. $m'(t)$ is the change in mass of rocket fuel in kg/s.

2. *Write down* ***what*** *relationship the differential equation is describing in common, no-nonsense language.*

   We see that acceleration, $h''(t)$, is related to $-10000m'(t)$. That means as we lose rocket fuel, we gain faster acceleration. We also have $50 + m(t)$ in the bottom of the fraction — that means we gain acceleration more slowly since we are dividing by this quantity. We also lose 9.8 additional units of acceleration.

   The second equation is much simpler — it just says we are losing rocket fuel at a rate of 0.1 kg/s.

3. *Explain* ***why*** *those relationships seem to make sense.*

   This is more complicated, but these relationships do make sense. For example, the 9.8 units of acceleration lost are due to gravity. It makes sense that we gain acceleration as we lose fuel, since we burn fuel to make the rocket go faster. Finally, dividing by $50 + m(t)$ is to account for the fact that the heavier the rocket is, the slower it will accelerate.

Chapter 48

HOMEWORK: UNDERSTANDING DIFFERENTIAL EQUATIONS

1. For each differential equation below, do the following steps.

    - Describe what each variable or function is measuring (if possible at this stage), and give correct units.
    - Describe what the equation is saying. Use phrasing like "If such-and-such is big, than such-and-such grows faster."
    - Explain why the relationships from the previous bullet point makes sense in terms of the story or physical situation.

    a. Let $T(t)$ be the temperature of a cooling object in degrees Celsius, and let t be measured in seconds. Newton's law of cooling state that $T'(t) = -k(T(t) - T_{\text{air}})$. Here T_{air} is the ambient air temperature.
    b. Let $H(t)$ be the height of a mountain measured in meters over a long period of time (t measured in millions of years). Suppose $H(t)$ satisfies the differential equation $H'(t) = -k(H(t))^{1/3}$.
    c. Let $y(t)$ be the fish population in a lake being harvested at rate H fish per year. Suppose $y(t)$ satisfies the differential equation $y'(t) = ky(t) - (m + cy(t))y(t) - H$. Here, $ky(t)$ represents the birth rate, $(m + cy(t))y(t)$ the natural death rate, and H the harvest rate.

2. Skim through the article "Campus drinking: an epidemiological model" by J. L. Manthey, A. Y. Aidoo & K. Y. Ward. You're not going to understand the whole article — that's okay! But let's try to figure out bits and pieces of it.

    Here is their first differential equation from secion 2 of the article.
    $$\frac{dN}{dt} = \eta - \eta N - \alpha NP + \beta S + \epsilon P$$

    1. What does the variables N, S, and P represent?
    2. In the first differential equation, what terms represent college students transitioning to drinking more? Which one represent college students transitioning to drinking less?
    3. What is a main conclusion of the article?

CHAPTER 49

INITIAL VALUE PROBLEMS

Often a differential equation has many solutions. Consider the population equation

$$P'(t) = 0.03P(t)$$

We saw in the last section that $P(t) = e^{0.03t}$ solves this differential equation. However, $73e^{0.03t}$ also solves this differential equation:

$$\begin{aligned} P'(t) &= 73e^{0.03t} \cdot 0.03 \\ &= 0.03 \cdot 73e^{0.03t} \\ &= 0.03P(t) \end{aligned}$$

So $e^{0.03t}$ and $73e^{0.03t}$ both solve the differential equation $P'(t) = 0.03P(t)$. In fact, any function of the form $P(t) = Ae^{0.03t}$ solves this differential equation. $Ae^{0.03t}$ is called the *general solution*, and A is called a *free parameter*, since it can be anything that we like. However, sometimes there are certain conditions called *initial conditions* which specify what the free parameters must be (in which case it wouldn't be very free!). A differential equation with given initial conditions is called an *initial value problem*. I won't ask you to solve the differential equation fully in this book, but solving for the free parameters is very doable.

Example **Initial Value Problem**

Problem Solve $P'(t) = 0.03P(t)$ where $P(0) = 2050$. (Hint: The general solution is $P(t) = Ae^{0.03t}$).

In this case, we just need to specify what A, since it is the only free parameter. We see that $P(t) = Ae^{0.03t}$ and $P(0) = 2050$. Therefore, $Ae^{0.03 \cdot 0} = 2050$. Anything to the zero is 1, hence we see $A(1) = 2050$, so $A = 2050$. So the final answer is $\boxed{P(t) = 2050e^{0.03t}}$.

Example **Another Initial Value Problem**

Problem Verify $M(t) = Ae^{t^2/2}$ satisfies the differential equation $M'(t) = tM(t)$. Then solve for the free parameter if $M(0) = -4$.

We see that $M(t) = Ae^{t^2/2}$, and we can compute $M'(t)$ using the chain rule: $M'(t) = Ae^{t^2/2} \cdot (2t/2) = tAe^{t^2/2}$. But notice this exactly fits the differential equation:

$$\begin{aligned} M'(t) &= tM(t) \\ (tAe^{t^2/2}) &= t(Ae^{t^2/2}) \\ tAe^{t^2/2} &= tAe^{t^2/2} \end{aligned}$$

Thus we have verified the solution to the differential equation. To find the free parameter A, we use $M(0) = -4$, and see that $Ae^{0^2/2} = -4$, and hence $A(1) = -4$, and so $A = \boxed{-4}$. This completes the verification and solving for the free parameter.

Chapter 50

HOMEWORK: INITIAL VALUE PROBLEMS

---

1. Verify that the given solution to each differential equation is correct, and solve for the free parameter.

   a. Differential equation $f'(t) = f(t) + 3$, solution $f(t) = Ae^t - 3$, $f(0) = 4$.

   $$\begin{aligned} f'(t) &= f(t) + 3 \\ \frac{d}{dt}(Ae^t - 3) &= (Ae^t - 3) + 3 \\ Ae^t &= Ae^t. \end{aligned}$$

   If $f(0) = 4$, then $A = 7$.
   ans

   b. Differential equation $f'(t) = 2f(t) - 2$, solution $f(t) = Ae^{2t} + 1$, $f(0) = 0$.

   $$\begin{aligned} f'(t) &= 2f(t) - 2 \\ \frac{d}{dt}(Ae^{2t} + 1) &= 2(Ae^{2t} + 1) - 2 \\ 2Ae^{2t} &= 2Ae^{2t} + 2 - 2 \\ 2Ae^{2t} &= 2Ae^{2t} \end{aligned}$$

   If $f(0) = 0$, then $A = -1$.
   ans

   c. Differential equation $f'(x) = \frac{1}{f(x)+1}$, solution $f(x) = \sqrt{A + 2x + 1} - 1$, $f(0) = 4$.

$$\begin{aligned}
f'(x) &= \frac{1}{f(x)+1} \\
\frac{d}{dx}(\sqrt{A+2x+1}-1) &= \frac{1}{(\sqrt{A+2x+1}-1)+1} \\
\frac{d}{dx}(A+2x+1)^{1/2} &= \frac{1}{\sqrt{A+2x+1}} \\
\frac{1}{2}(A+2x+1)^{-1/2}\cdot 2 &= \frac{1}{\sqrt{A+2x+1}} \\
(A+2x+1)^{-1/2} &= \frac{1}{\sqrt{A+2x+1}} \\
\frac{1}{\sqrt{A+2x+1}} &= \frac{1}{\sqrt{A+2x+1}}
\end{aligned}$$

If $f(0) = 4$, then $A = 24$.
ans

d. Differential equation $f'(t) = (f(t))^2 + f(t)$, solution $f(t) = -\frac{Ae^t}{Ae^t - 1}$, $f(0) = 3$.

$$
\begin{aligned}
f'(t) &= (f(t))^2 + f(t) \\
\frac{d}{dt}\left(-\frac{Ae^t}{Ae^t-1}\right) &= \left(-\frac{Ae^t}{Ae^t-1}\right)^2 + \left(-\frac{Ae^t}{Ae^t-1}\right) \\
-\frac{(Ae^t-1)Ae^t - Ae^t(Ae^t)}{(Ae^t-1)^2} &= \frac{(Ae^t)^2}{(Ae^t-1)^2} - \frac{Ae^t}{Ae^t-1} \\
-\frac{(Ae^t)^2 - Ae^t - (Ae^t)^2}{(Ae^t-1)^2} &= \frac{(Ae^t)^2}{(Ae^t-1)^2} - \frac{Ae^t}{Ae^t-1}\cdot\frac{(Ae^t-1)}{(Ae^t-1)} \\
-\frac{-Ae^t}{(Ae^t-1)^2} &= \frac{(Ae^t)^2}{(Ae^t-1)^2} - \frac{(Ae^t)^2 - Ae^t}{(Ae^t-1)^2} \\
\frac{Ae^t}{(Ae^t-1)^2} &= \frac{(Ae^t)^2}{(Ae^t-1)^2} + \frac{-(Ae^t)^2 + Ae^t}{(Ae^t-1)^2} \\
\frac{Ae^t}{(Ae^t-1)^2} &= \frac{(Ae^t)^2 - (Ae^t)^2 + Ae^t}{(Ae^t-1)^2} \\
\frac{Ae^t}{(Ae^t-1)^2} &= \frac{Ae^t}{(Ae^t-1)^2}
\end{aligned}
$$

If $f(0) = 3$, then $A = 1.5$.
ans

Chapter 51

Growth and Decay

We've seen a differential equation pop up several times already, and it is the most common and simplest of all differential equations:

$$G'(t) = kG(t)$$

When k is positive, this is saying that G is growing at a rate proportional to the value of the function at any given point. As we've seen, population tends to follow this rule, but several other things do as well. When k is negative, this is saying that G' is decreasing at a rate proportional to its value, and this is true for several things as well. These are called the *growth* and *decay* equations respectively.

And there is a simple solution to the differential equation $G'(t) = kG(t)$. It is $G(t) = Ae^{kt}$. Let's see some examples

Example **Decay**

Problem A radioactive isotope decays at a rate of 0.003 times its mass in grams per day. Initially, a sample contains 40 grams of the isotope at $t = 0$.

1. How much of the isotope will there be left at $t = 365$?
2. At what time will there be 1 gram left?

Let $I(t)$ be the mass of isotope in grams at time t. Thus, $I(0) = 40$, since we start with 40 grams. Since the isotope decays at a rate of 0.003 times its current mass, we see that $I'(t) = -0.003 \cdot I(t)$. The negative is in there because it is a decay rate — the amount of isotope is going down. We know the solution to a differential equation like this is

$$I(t) = Ae^{-0.003t}$$

Since $I(0) = 40$, we also have $I(0) = Ae^{-0.003(0)} = Ae^0 = A$, and hence $A = 40$. Our equation for the mass of the isotope is now $I(t) = 40e^{-0.003t}$

From here, we can now tell how much isotope will be left at $t = 365$. We plug in $t = 365$ and have

$$I(365) = 40e^{-0.003(365)} \approx \boxed{13.38 \text{ grams}}.$$

This solves part (1).

To find out when there will be 1 gram left, we solve

$$\begin{aligned} 1 &= 40e^{-0.003t} \\ \frac{1}{40} &= e^{-0.003t} \\ \ln\left(\frac{1}{40}\right) &= -0.003t \\ \frac{1}{-0.003}\ln\left(\frac{1}{40}\right) &= t \end{aligned}$$

Simplifying this, we see $t \approx \boxed{1229.62 \text{days}}$ or a little over 3 years. This solves part (2).

Chapter 52

HOMEWORK: GROWTH AND DECAY

1. Money that is compounded continuously follows the differential equation $M'(t) = rM(t)$, where t is measured in years, $M(t)$ is measured in dollars, and r is the rate. Suppose $r = 0.05$ and $M(0) = 1000$.

   a. What is a function that satisfies this initial value problem?
   We know from class that this is an exponential $M(t) = 1000e^{0.05t}$.
   ans
   b. How much money will there be at year 30 (i.e. $t = 30$)?
   $4481.69
   ans
   c. When will there be 2000 dollars?
   13.86 years.
   ans

2. The mass of bacteria on a deceased animal follows the equation $M'(t) = 0.1M(t)$, where $M(t)$ is measured in grams and t is measured in hours.

   a. If $M(0) = 1$, what is a function that satisfies this initial value problem?
   $M(t) = e^{0.1t}$
   ans
   b. How much bacteria will there be at $t = 24$?
   11.02 grams
   ans
   c. When will there be one kilogram of bacteria?
   2 days, 21 hours
   ans

3. For a cooling object outside in 0° degree weather, temperature decreases according to the differential equation $T'(t) = -0.05T(t)$, where t is measured in minutes and $T(t)$ measured in Fahrenheit.

   a. If the temperature is initially 72°, what is the function that satisfies this initial value problem?
   $T(t) = 72e^{-0.05t}$
   ans
   b. What is the temperature after 1/2 hour?
   16.06 degrees

ans

c. At what time did the object reach the freezing point of water?
   Approximately 16 minutes
   ans

CHAPTER 53

EXPLORING GRAPHS OF DIFFERENTIAL EQUATIONS

In this section we will focus on differential equations that model climate change. Now compared to sophisticated climate models used by climate scientists, this is just a toy model and doesn't cover all the complexities of real climate. However, it does show how feedback loops can amplify the effects of climate change.

Photo by Ian Barbour

In terms of global earth temperature, what greenhouse gas is responsible for trapping the most heat? You may be surprised to learn that it is H_2O, not CO_2. Water vapor is a more effective greenhouse gas than carbon dioxide, and there is a lot more of it too. So why don't more people talk about H_2O in regards to global warming? It's because the amount of H_2O is not a driving force behind climate change. The H_2O is dependent on temperature to begin with. Since you can't increase the H_2

O without increasing temperature, you can't use H_2O to increase temperature. However, that doesn't mean H_2O can be ignored either.

Consider the following set of differential equations. $T(t)$ is the global temperature, $W(t)$ is the amount of water, and $C(t)$ the amount of carbon.

$$\frac{d}{dt}T = c - dT + eW + fC$$
$$\frac{d}{dt}W = a(gT - W)$$
$$\frac{d}{dt}C = b$$

Problem Follow the steps from the section on understanding differential equations to understand what this differential equation is saying

Okay, let's go through the steps.

1. *Understand what each variable is measuring with correct **units**.*

   There are no units listed in the problem, so we will make up some reasonable guesses for what they are. Anyway, t is a measure of time as usual, we could say measured in years after some starting date. $T(t)$ is the global temperature, we will say measured in Celsius. $W(t)$ is the average concentration of water vapor globally at time t, and $C(t)$ is similar for CO_2. We will assume $W(t)$ and $C(t)$ are measured in parts per million, or ppm. Then we have $\frac{d}{dt}T$ is how fast the temperature is changing globally (in degrees Celsius per year), $\frac{d}{dt}W$ is the change of water concentration (in ppm per year), and $\frac{d}{dt}C$ is the change in CO_2 (in ppm per year). Note that while I "guessed" the units for t, $W(t)$ and $C(t)$, the units for a derivative are forced from those choices, so I can't just make up new units for say $\frac{d}{dt}W$. See this chapter on interpreting the derivative for more information.

   So what are a, b, c, d, e, f, g? Well, we'll talk about these more in the next step but they are basically parameters that relate how much of an effect various quantities have on each other.
2. *Write down **what** relationship the differential equation is describing in common, no-nonsense language.*

   Let's start with the simplest equation: $\frac{d}{dt}C = b$. This is just saying that carbon is increasing (or decreasing) at a constant rate b. This tells us what b is as well; assuming b is positive, it's how fast we're putting carbon into the atmosphere.

   The next simplest equation is $\frac{d}{dt}W = a(gT - W)$. Ignoring a and g for a second, this tells us that water vapor increases at a rate that looks like the difference between temperature and water vapor. So a hot and dry atmosphere will tend to get wetter, while a cool or damp atmosphere might mean a negative difference so it would get less wet. The values of a and g affect the severity of these trends.

   Finally, let's look at the more complicated equation: $\frac{d}{dt}T = c - dT + eW + fC$. Let us break that down into two pieces, starting with $c - dT$. Assuming c is positive, the c is basically a constant source of temperature increase. However, as the temperature increases, we see that the $-dT$ term will tend to cool things off. We will see in the next step what these terms really mean. But for now, let's move on to $+eW + fC$. This is saying that the bigger W and C are, the hotter the earth will get.

3. *Explain **why** those relationships seem to make sense.*

   We just got done thinking about the equation $\frac{d}{dt}T = c - dT + eW + fC$, so let's start there. Let's focus on $c - dT$. Why are these terms here influencing temperature? Well, we said c was a constant source of temperature increase — does that ring a bell? Yes, it's the sun! Ignoring other effects, we will continue to absorb solar energy until we are hotter than Venus. So hopefully there is something that will cool us off. In this equation it is the term $-dT$. Why is this term here? Well, it turns out the hotter the earth gets, the more heat energy will be radiated away. We don't generally see this energy with our eyes, because it is infrared, but it's there. This $c - dT$ is the basic energy balance that determines the temperature of the earth. A similar equation would hold true for any planet.

   However, to this, we add $+eW + fC$. Why? That's right — water and carbon are greenhouse gases, so the more W and C, the hotter the earth will tend to get. Now technically, these are really just absorbing that infrared energy from the earth, so they are not actually separate from the $-dT$ term, but for simplicity I've just listed them as separate terms. This is one of many simplifications in this model that a real climate model would correct. It's important to note at this point that, even though carbon is the driver behind global warming, it's actually the water that is the better greenhouse gas and much more prevalent. Because of this, the e term should be much larger than the f term.

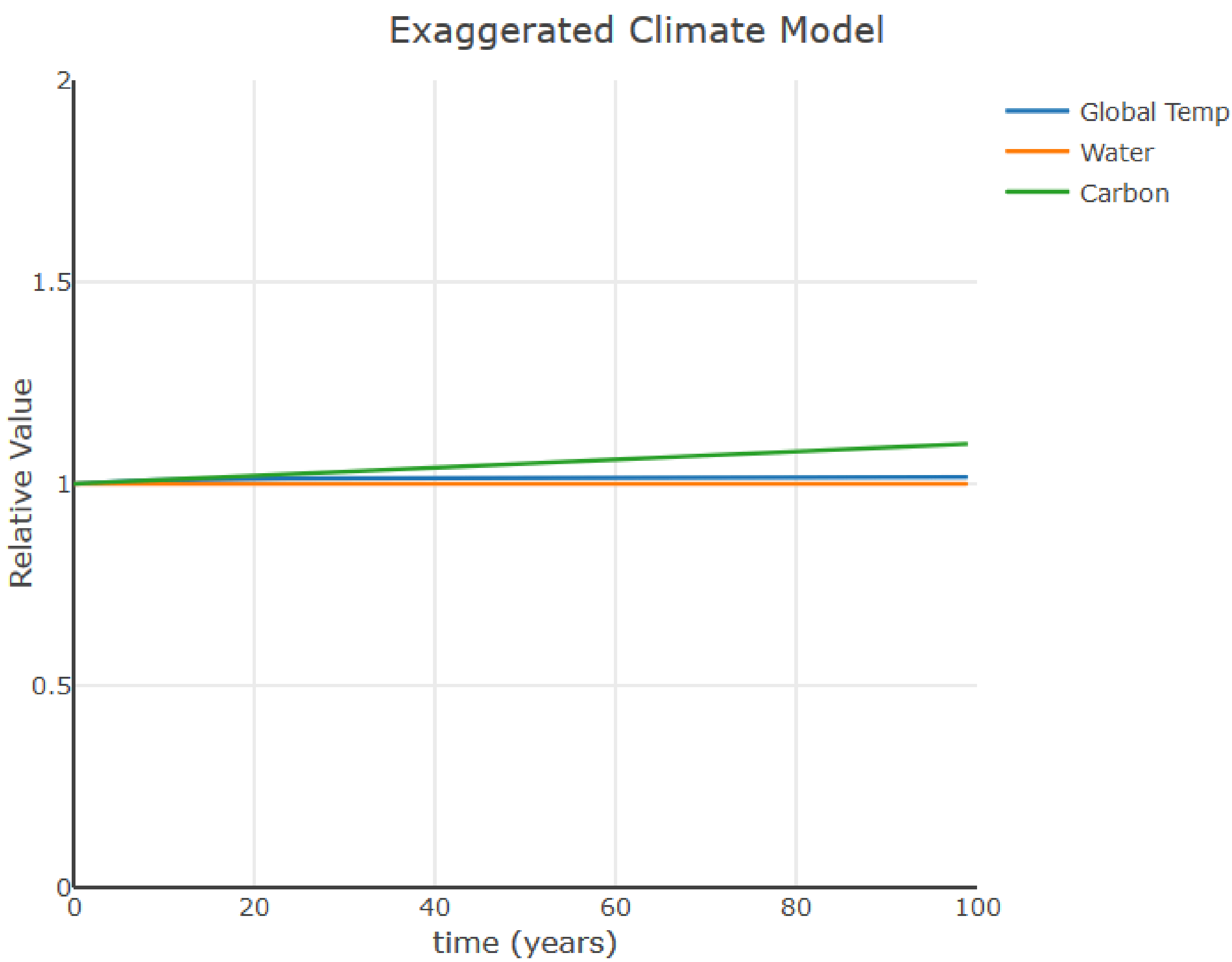
   Let's focus now on the $\frac{d}{dt}W = a(gT - W)$ equation. We already said that this equation implies that hot and dry air will tend to absorb more water, while cool and damp air would lose water. This makes sense since hot air just holds more water. Also, the hot air will heat the oceans causing evaporation.

   Finally, why do we have the equation $\frac{d}{dt}C = b$? Well, the amount of carbon in the atmosphere is a very complex system. But for simplicity we can say it is increasing at a constant rate b, due mostly to humanity burning fossil fuels.

Okay, now we understand some things about these differential equations, what do they tell us? For this, we will turn to computers to create some graphs for us. These graphs are all solutions to the differential equations. And while these graphs won't perfectly depict reality, they will show us some aspects of climate change you might not realize. We will use this website that graphs the solution for us. It looks like this:

DiffEQ Grapher

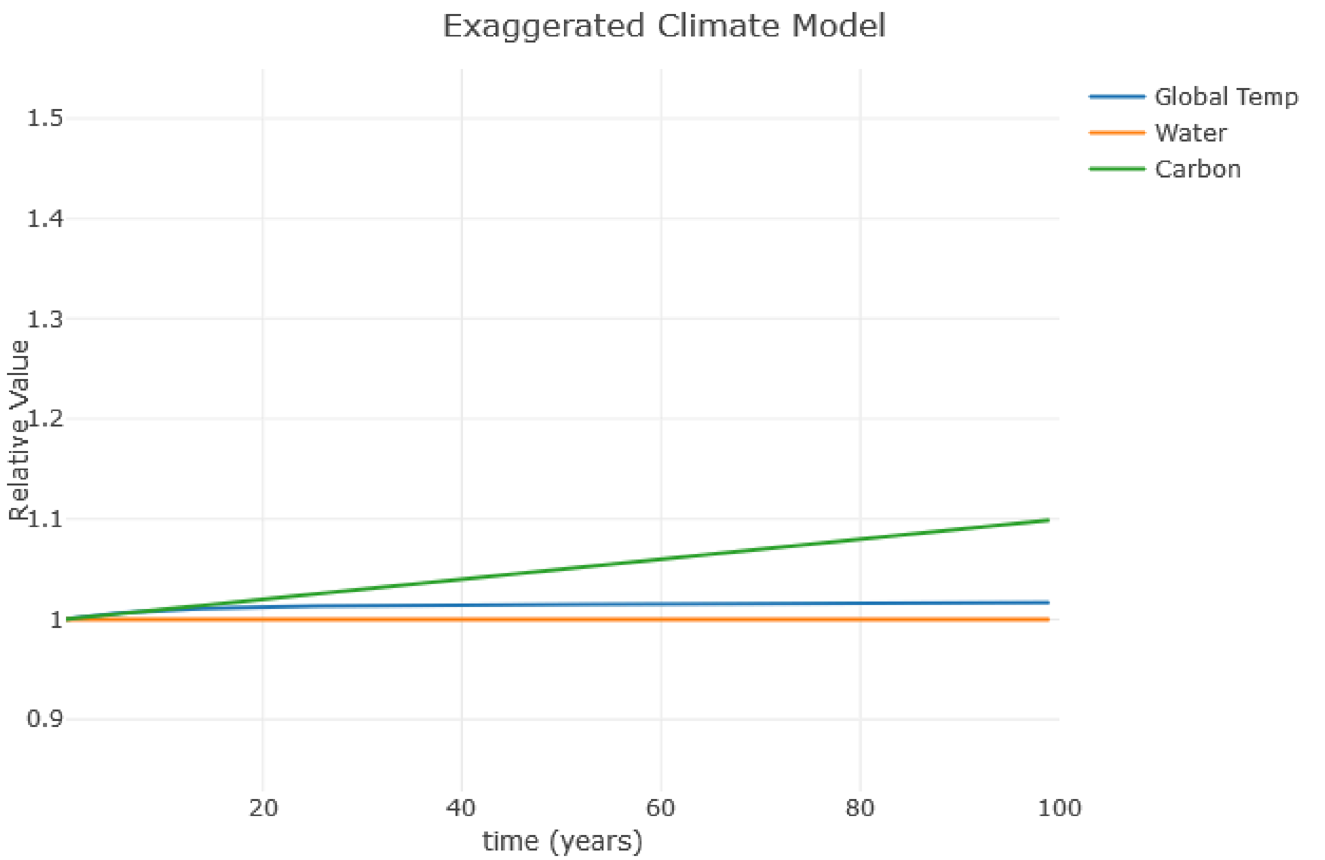
d/dt T = c-d*T+e*W+f*C (initial value: 1, name: Global Temp)
d/dt W = a*W*(T-W) (initial value: 1, name: Water)
d/dt C = b (initial value: 1, name: Carbon)
d/dt = (initial value: , name:)



a = 0
b = 0.001
c = 0.0106
d = 0.1106
e = 0.0973
f = 0.0041

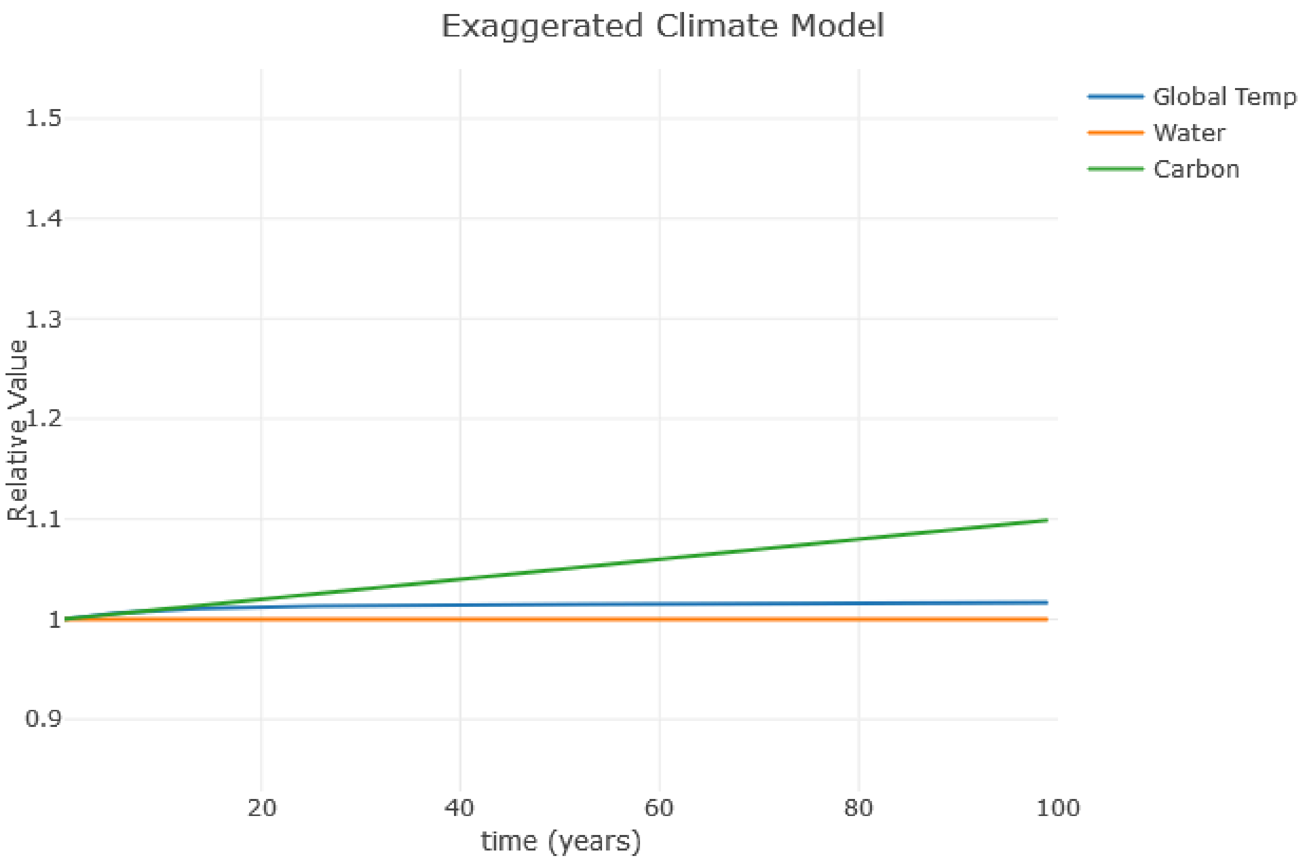
Max t value: 100
Number of Iterations: 100
Max y value: 2

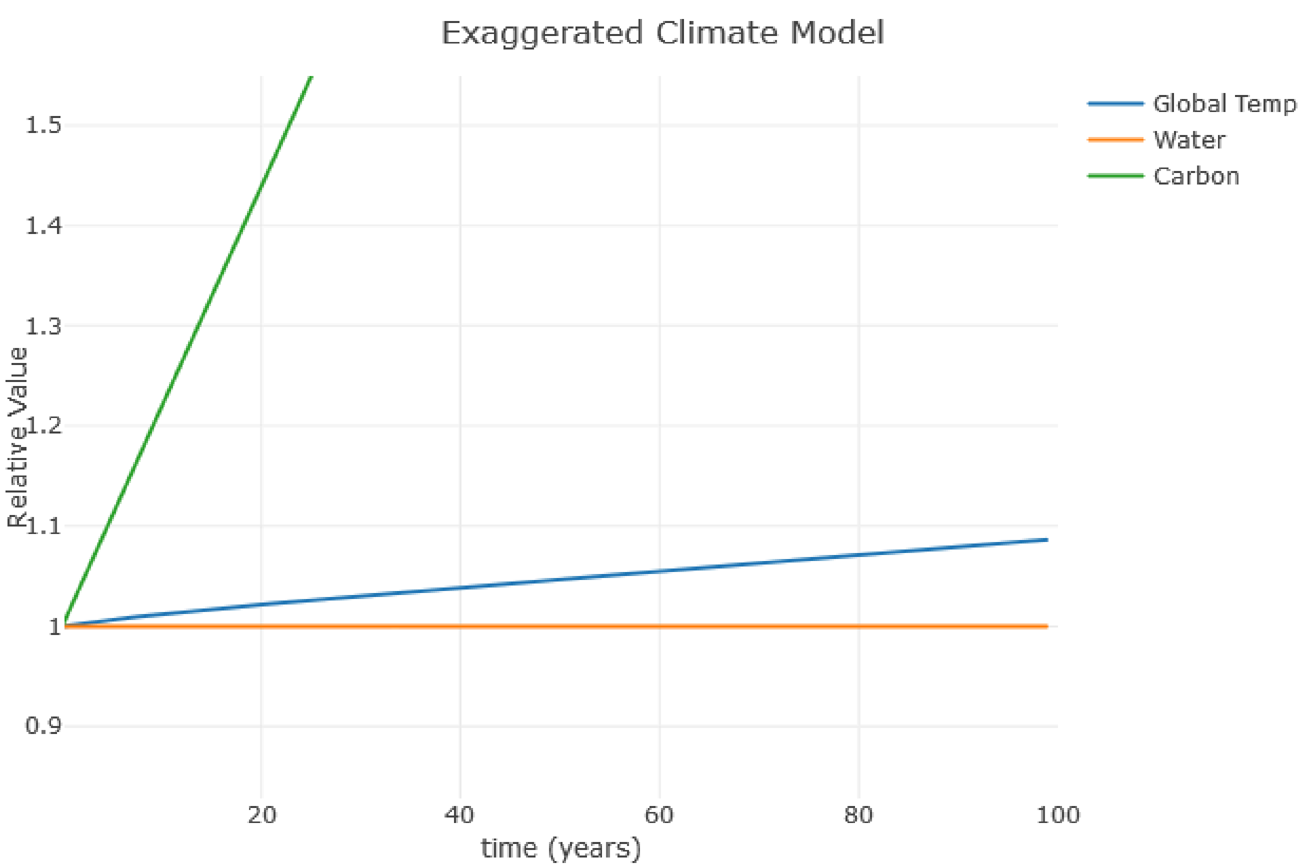
Let's take a closer look at that graph. Note we have $a = 0$, $b = 0.001$, $c = 0.01$, $d = 0.1$, $e = 0.1$, and $f = 0.004$. Notice that f is much smaller than e, reflecting how much less of an effect carbon has compared to water.



Okay, not a lot happening yet. But you can start to get familiar with the graph. First, you'll note that Global Temperature, Water, and Carbon levels are all around 1. Note that does not mean the temperature is 1° C, or that Carbon is 1 ppm. Instead this represents some sort of "relative" value of these quantities compared to normal. This makes it easier to put these things on the same graph. So 1 represents normal, 2 would be twice normal, 0.5 would be half of normal, and so on.

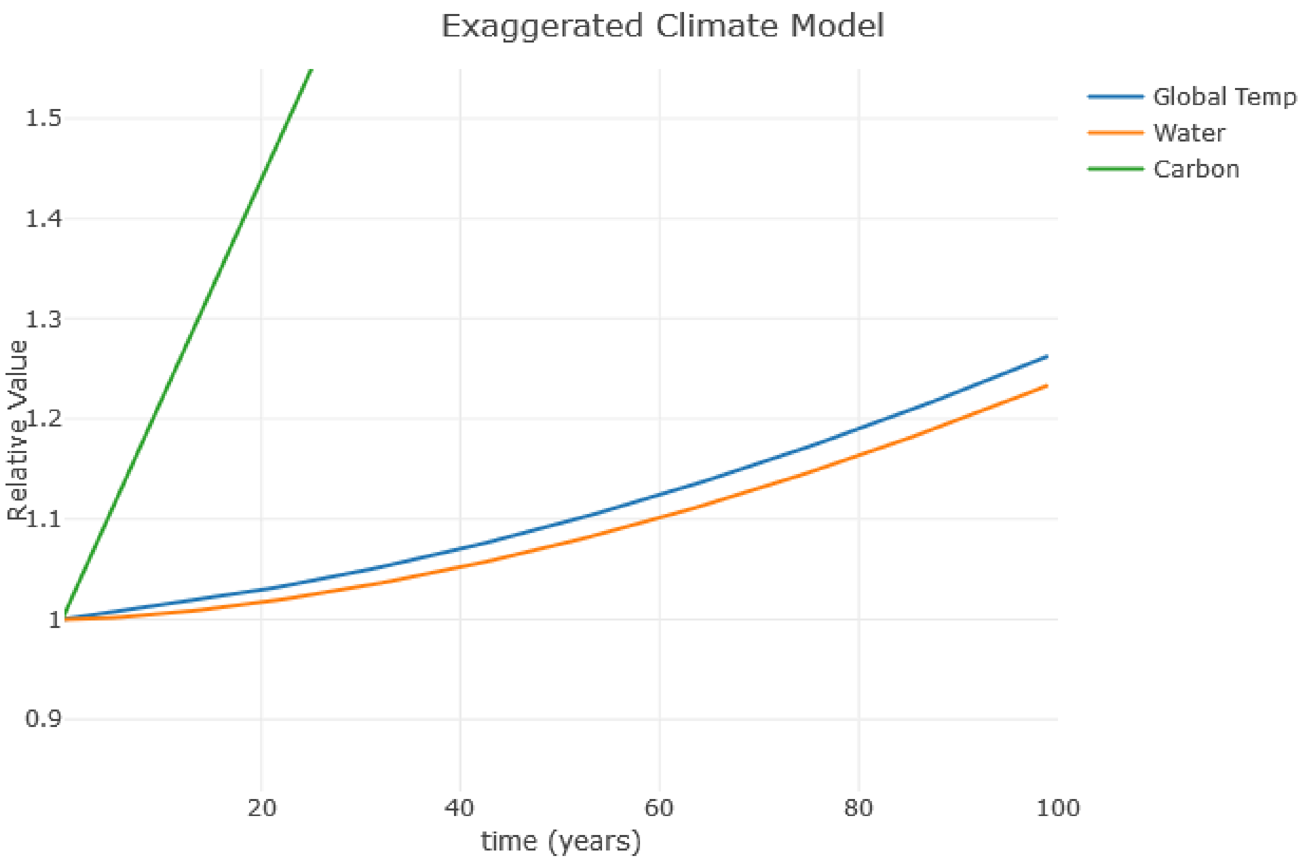
The amount of carbon in the atmosphere related to the b value. So if I increase this from 0.001 to about 0.02, we see the change in the graph below.





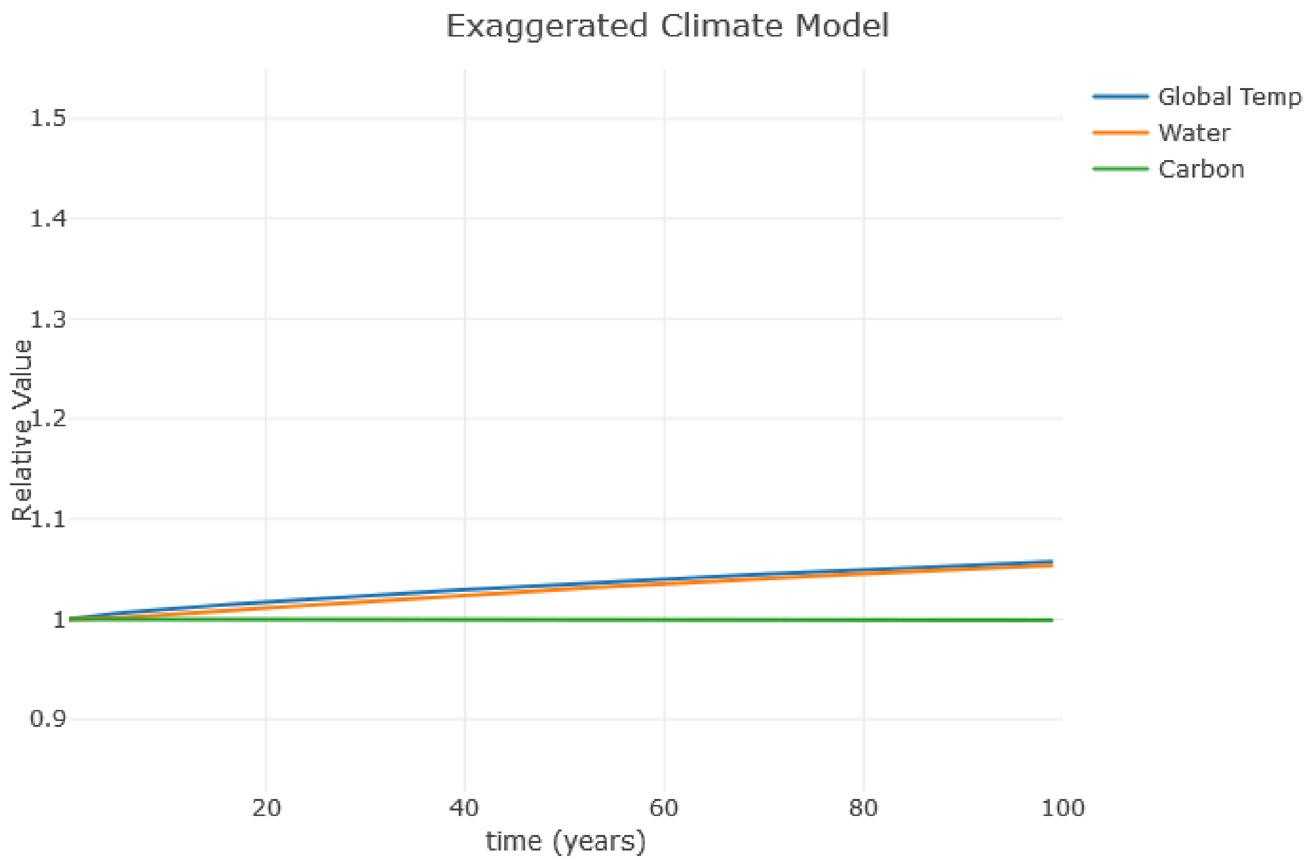
Now the carbon is taking off, but not much is happening to the global temperature. Why is this?

Well, notice we set $a = 0$. This has the effect of keeping the water value constant, even though in reality it would increase as the temperature increases. So let's see what happens when we set $a = 0.1$:



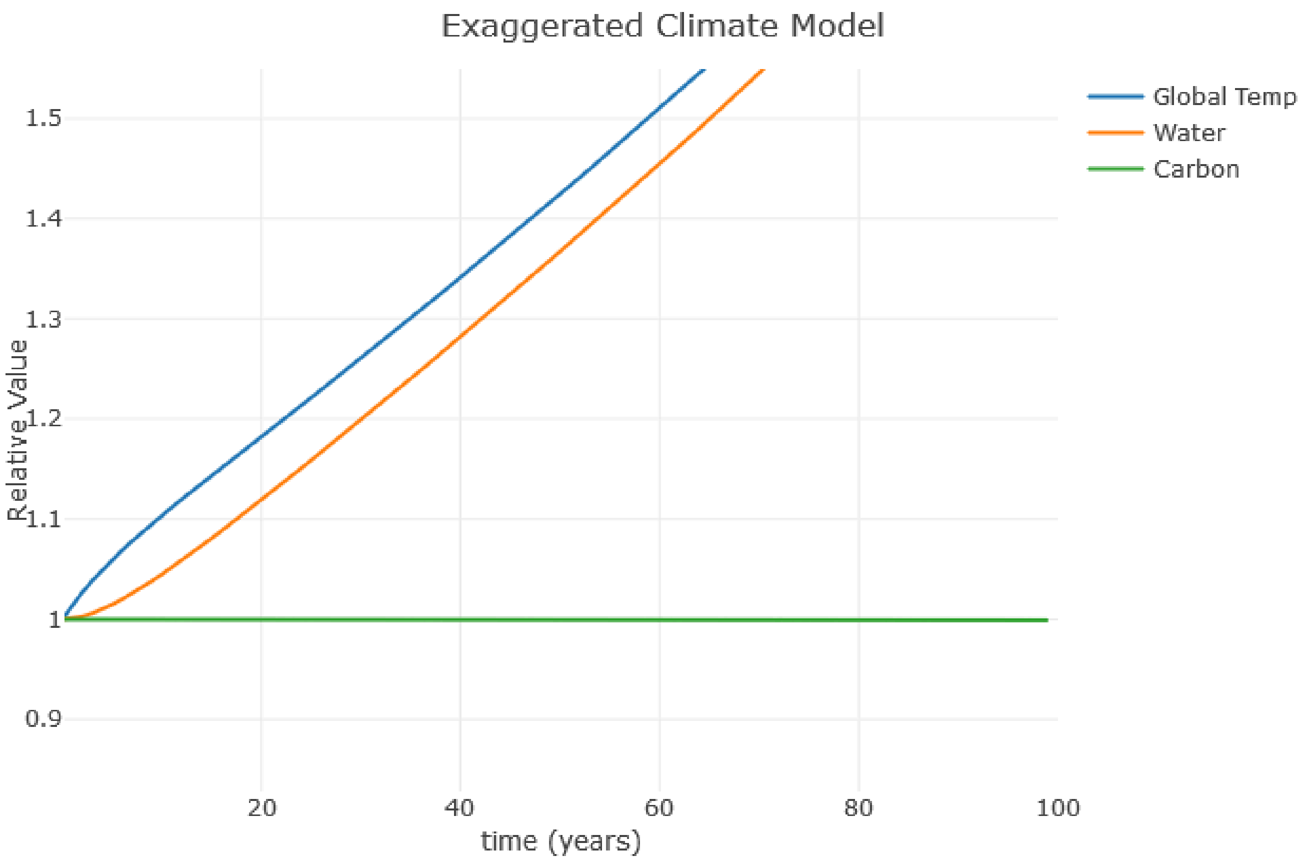
Now we can see how the temperature increases the water vapor, which furthers increases the temperature, which further increases the water vapor, and so on. This is a classic example of a climate feedback loop. The end result is a much higher global temperature with the same basic increase in carbon. This is consistent with more advanced models, which show that water roughly doubles the effect of carbon. Note that I don't expect a 25% increase in temperature over the next 100 years – this is exaggerated to illustrate the relationships.

A couple more things: Suppose we don't increase carbon anymore, but still have that water vapor effect. What happens? Here, I've left a at 0.1, but reduced b to 0.



Clearly, this is much better than when we were increasing carbon so fast. However, notice there is still an increase in temperature. That's because these feedback loops continue to operate even after we stop increasing carbon. As a result, scientists expect global temperatures to rise for several decades even if we manage to become carbon neutral as a planet.

Finally, the worst-case scenario is as follows. It has the same settings as the previous example, just with e (the effect of water vapor) increased 10%



This is the "runaway greenhouse effect", where a feedback loop gets out of control, and even with no additional carbon the temperature increases to the point where the seas boil away. Most scientists think this isn't possible for earth even with large amounts of additional carbon, but with the right conditions this sort of thing is possible. Scientists think this is what happened to Venus hundreds of millions of years ago.

Please feel free to play around with the model yourself!

Chapter 54

PROJECT: MODELLING WITH DIFFERENTIAL EQUATIONS

Purpose: To be introduced to the concept of modeling with differential equations

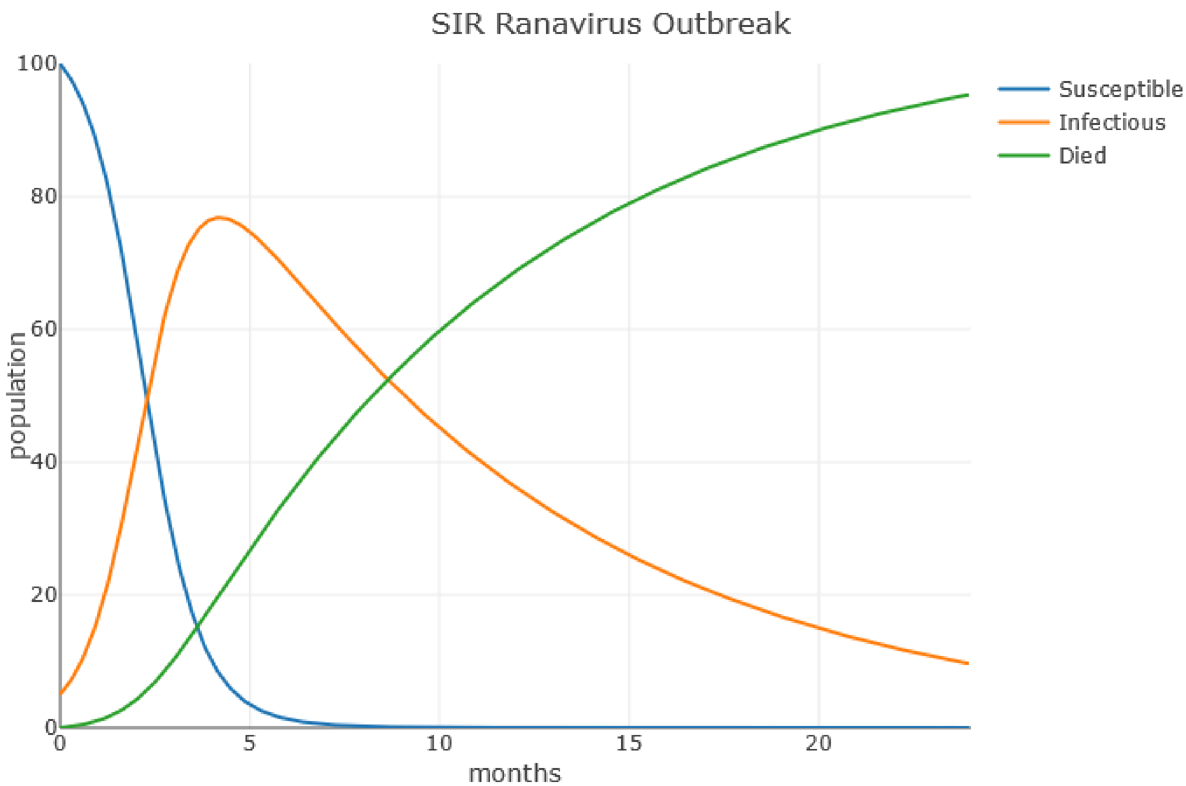
In this project, you'll choose a well-known differential equation model from biology, try to understand the differential equations, and explore graphical solutions.

Choose one of the following topics:

- SIR ranavirus model. Ranavirus is a disease that affects reptiles, amphibians, and fish; "Ranavirus is believed to be the cause of several recent mass mortality events in amphibian populations across the globe" (link). Given a population, let $S(t)$ be the number of frogs suseptible to ranavirus, let $I(t)$ be the number of frogs currently infected with the disease, and let $R(t)$ be the number of frogs that have died. Note that at any time, $S(t) + I(t) + R(t)$ is the equal to the total population.Then

$$\frac{dS}{dt} = -aS(t) \cdot I(t)$$
$$\frac{dI}{dt} = aS(t) \cdot I(t) - bI(t)$$
$$\frac{dR}{dt} = bI(t)$$

  Here, a and b are unknown parameters that affect the dynamics of this problem. Click here for the DiffEQ grapher for the Ranavirus SIR model.

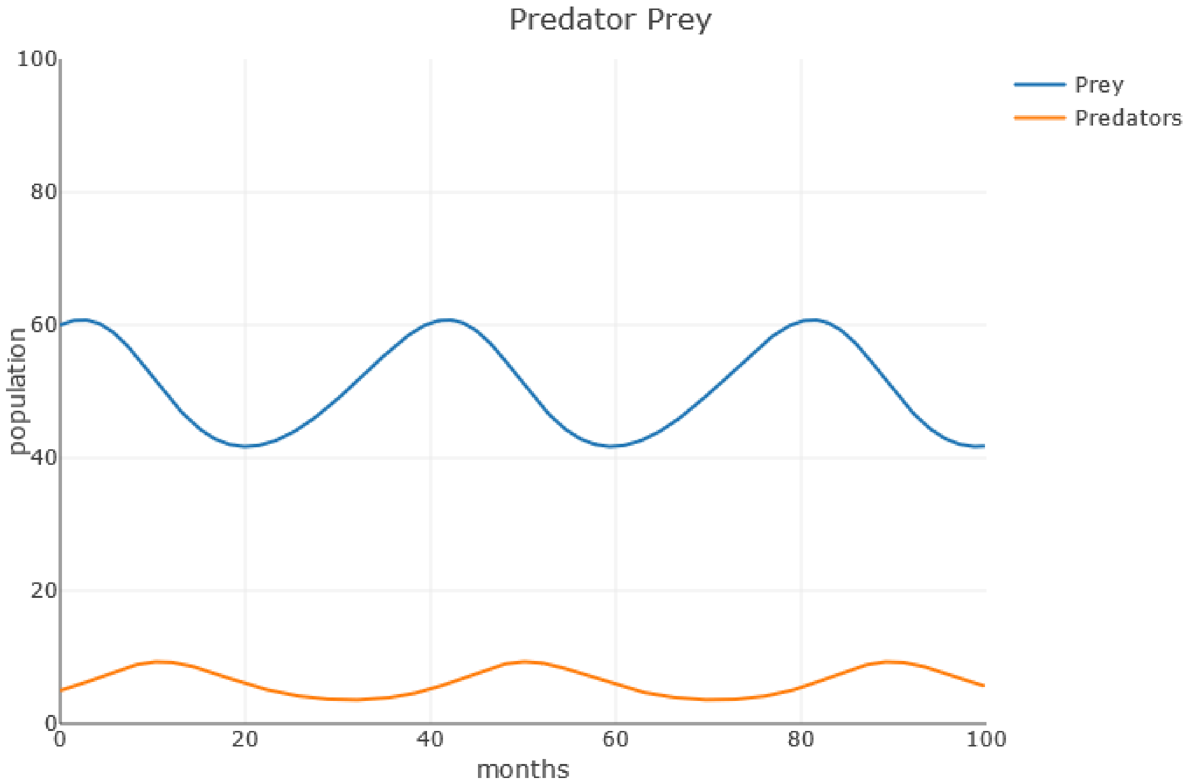


- Lotka-Volterra equations for simple modeling of predator and prey dynamics, such as the moose and wolf populations in Isle Royale National Park (link). Given M is a population of prey (Moose), and W is the population of predators (wolves), we have

$$\frac{dM}{dt} = aM(t) - bM(t) \cdot W(t)$$
$$\frac{dW}{dt} = -cW(t) + dM(t) \cdot W(t)$$

Here, a, b, c, and d are unknown constants, but they are parameters that affect the interaction between the two species.

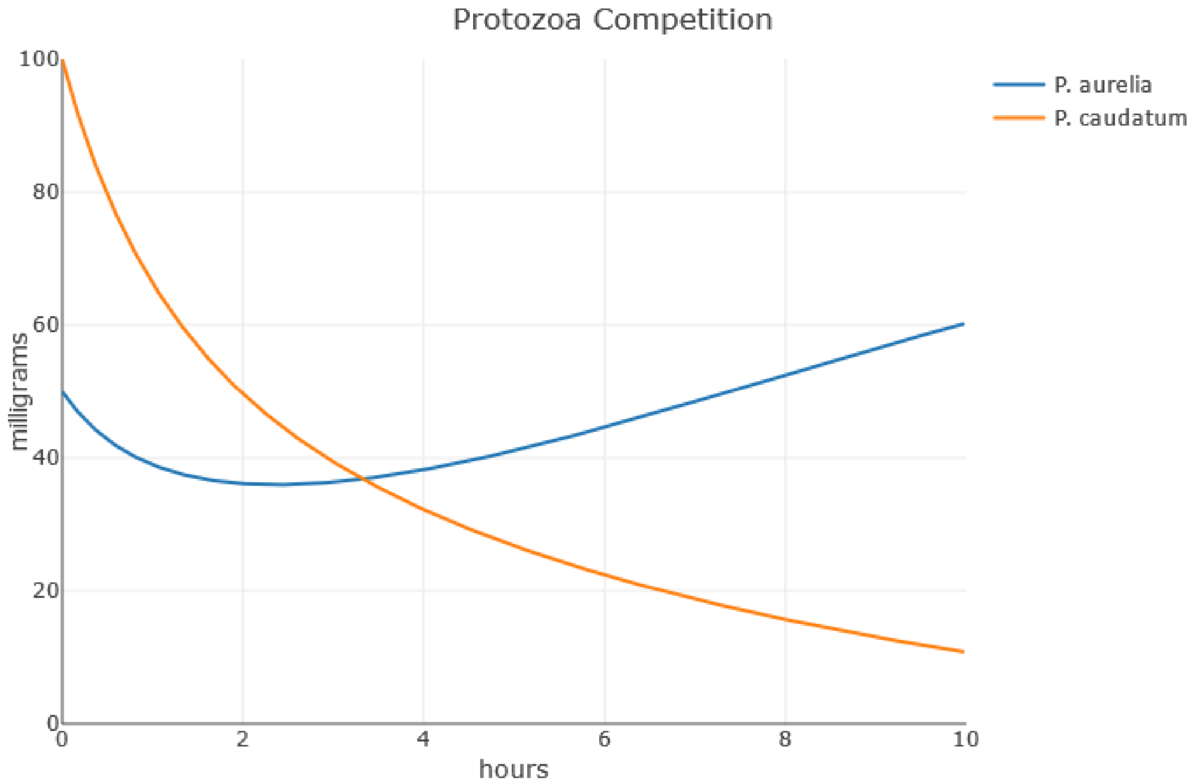
Click here for the DiffEQ grapher for the Lotka-Volterra equations.



- Protozoa Competition: Paramecium aurelia and Paramecium caudatum are two species of single-celled protozoa, which were studied by G.F. Gause when he formulated his famous Competition exclusion principle (link). Let $A(t)$ be milligrams of Paramecium aurelia, and $C(t)$ be milligrams of Paramecium caudatum. Suppose A and C satisfy:

$$\frac{dA}{dt} = aA(t) - b(A(t) + C(t))A(t)$$
$$\frac{dC}{dt} = cC(t) - d(A(t) + C(t))C(t)$$

  Here, a, b, c, d are the parameters that affect this problem.
  Click here for the DiffEQ grapher for the Protozoa equations.



Once you've chosen a topic, here is what to do:

1. Follow the steps from the section on understanding differential equations to understand what the differential equation is saying.
2. Using the supplied "DiffEQ" webpage, explore graphical solutions to the differential equation.
    a. What do the parameters a, b, etc., represent?
    b. What setting for the parameters create realistic looking graphs?
    c. How can you "break" the model and create unrealistic graphs?
    d. What are some different ways to create a relatively good outcome (that is, few people get the disease)?
3. Try to think of something this simple model doesn't take into account.
    a. How could you modify the differential equations to account for this factor?
    b. Using the DiffEQ webpage, explore graphical solutions to your new differential equations. Do you feel like you were successful in implementing the new factor? What lessons can be learned from the graphs?

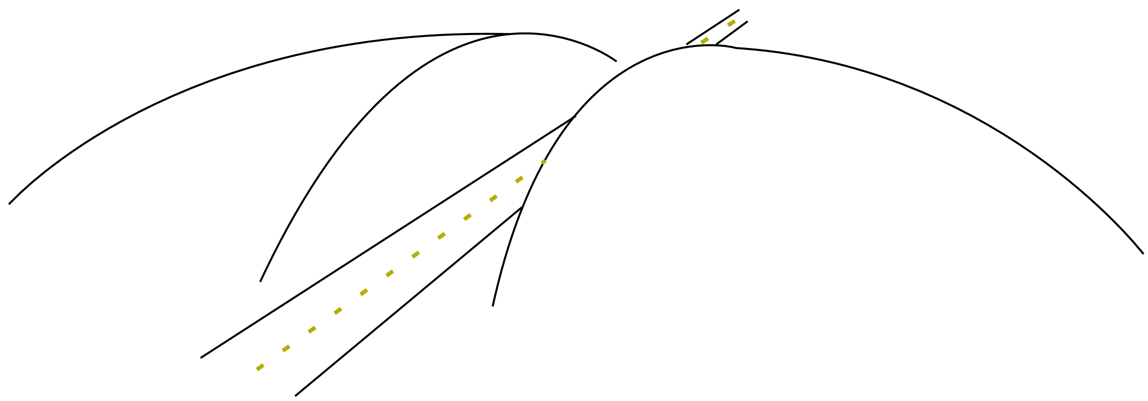
PART VI

INTUITION FOR INTEGRATION

CHAPTER 55

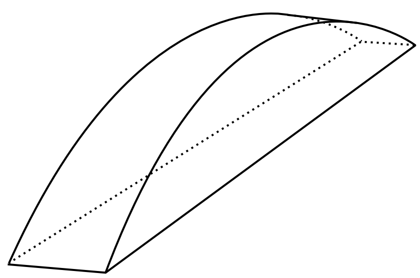
INTRODUCTION TO INTEGRALS

Suppose construction workers needed to remove dirt in a hill in order to run a road though it. The swath needed to be excavated is 10 meters wide, and the hill is parabola shaped. Here is a picture:

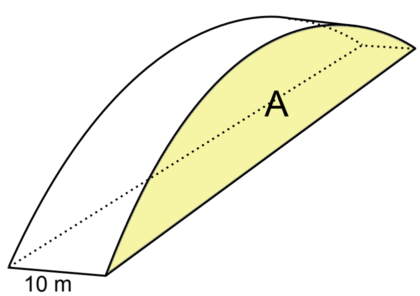


How much dirt needs to be removed? What do we need to know to solve the problem?

That's right — we need to know the *volume* of the region we're excavating. Here is a picture of the region we are removing:

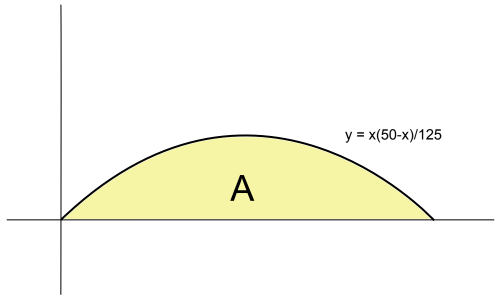


Such an object is called a "prism" — it's basically two dimensional shape that has been given a little thickness. Finding the volume involves finding the area of the shape times its thickness:

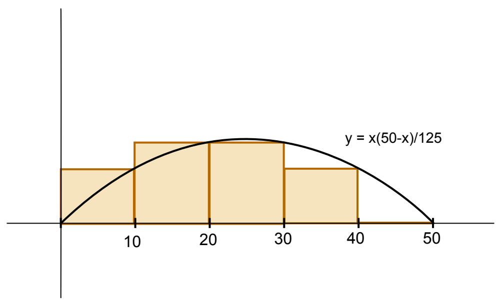


In this case, the area is labeled A in the diagram, and the width is 10. So the amount of dirt being hauled away is $10A$. But what is A? How do we find it? That is where integration comes in.

A is a parabola like shape, but let's say it is bounded on top by the function $y = x(50 - x)/125$. Thus you get a picture like this:



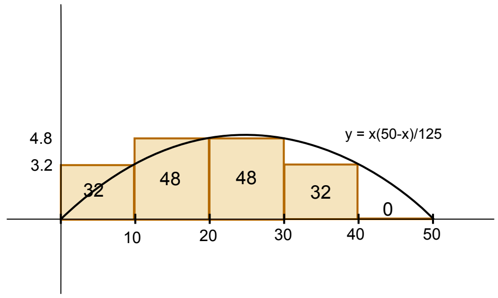
How do we find the area of a curved shape like this? Well, we don't know how to find the area of an arbitrary curved shape. But we DO know how to find the area of a rectangle: it is height times width. So let's use rectangles:



So it's not a great approximation yet, but bear with me. The area of the boxes does approximate the area under the curve to some extent. But how do we figure out the area of the rectangles? Well, that's height times width, and we can see that the width of each box is 10. So we just need the height of each rectangle.

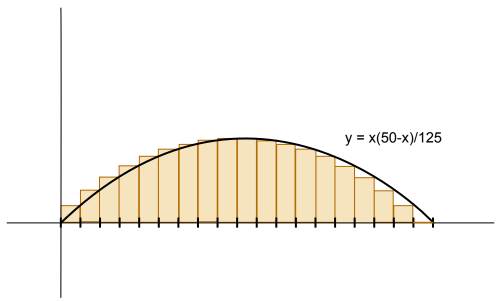
When I drew each rectangle, I made it so that the right side of the rectangle was exactly the height of the function. The height of the function is something we can find! That's because we know the hill follows the function $y = x(50 - x)/125$. The height of the hill, and the height of the rectangles can then be found. For example, the height of the first rectangle is just plugging $x = 10$ into our function. So the height is $y = 10(50 - 10)/125 = 400/125 = 16/5 = 3.2$. The height of the second rectangle is $y = 20(50 - 20)/125 = 600/125 = 24/5 = 4.8$. The third rectangle has height $y = 30(50 - 30)/125 = 600/125 = 4.8$, the fourth rectangle has height $y = 40(50 - 40) = 400/125 = 3.2$, and the fifth has height $y = 50(50 - 50)/125 = 0/125 = 0$.

If we take these heights and multiply by the width, we get the area of each rectangle.



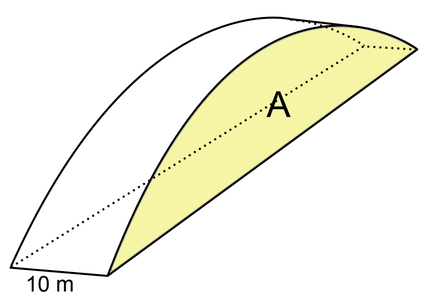
Adding up all the area from all the rectangles, we have $32 + 48 + 48 + 32 + 0 = 160\,\mathrm{m}^2$. This is an approximation of the area.

Now you might say that this approximation might not be very good, and you're correct. It certainly gives the rough idea, which might be all that we need. But if we need a very precise answer, 160 isn't good enough. So what can we do? Let's use more rectangles!



The idea of using rectangles like this to get better and better approximations of area is called a Riemann Sum. Now, we could find the height and width of each of those rectangles, but it would be a bit tedious. Since mathematicians are lazy, we often have a computer do the work for us. For example, website "MathWorld" has a Riemann Sum calculator: http://mathworld.wolfram.com/RiemannSum.html. Using this calculator, we can see that when using 20 rectangles, the sum becomes 166.25. This is called *numeric integration* (If you use more and more rectangles, you can actually find a very precise answer of 166.667 — this is how the program computes the "actual area". However, this is still an approximation, not an exact value.)

What does this have to do with the dirt problem? Remember, we were trying to find the volume of this slice:



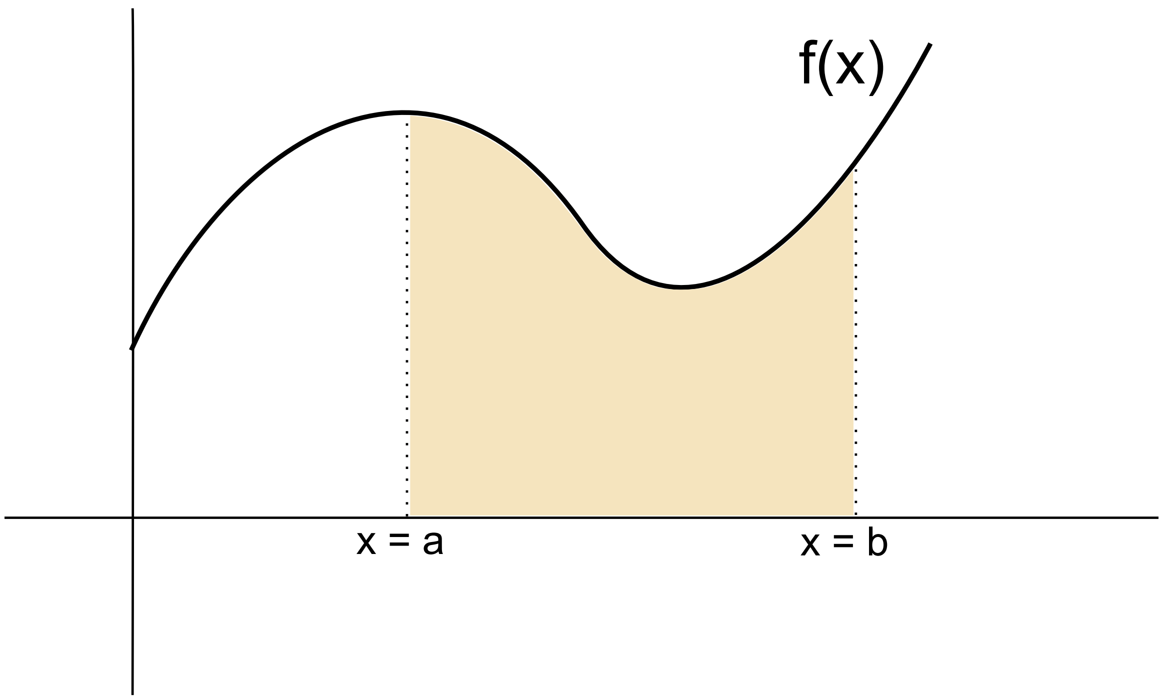
What we have done with the Riemann sums is find $A = 166.667$. To find how much dirt is hauled away, we just need to multiply by the width of the prism, which is 10 m. Hence 1666.67 cubic meters of dirt must be hauled away.

Chapter 56

NUMERIC INTEGRATION TECHNIQUES

This process of finding the area underneath a curve is used for a lot more than finding how much dirt needs to be hauled away — in fact, it's vital to many physics, engineering problems, and it even crops up in environmental science and biology problems. Let's do some more examples to get a feel for how it works, and to introduce the notation for it.

Given a function $f(x)$, the area under the curve from $x = a$ to $x = b$ looks something like this:

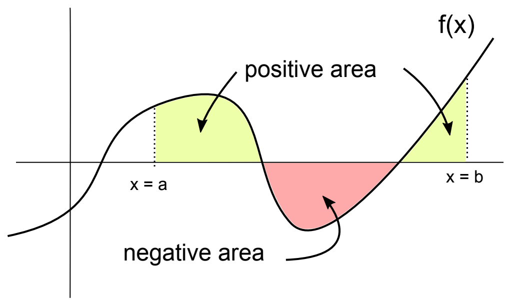


This area is denoted by mathematicians as

$$\int_a^b f(x)dx.$$

Here, the a and b indicate the left and right boundaries of the area we are interested in. The $\int$ and the dx you can think of as just part of the notation for now, though they relate to how mathematicians write finite sums (with the integral being a sort of infinite sum). Sometimes this is called a *definite integral* to separate it from an indefinite integral. A definite integral is an area under a curve, and indefinite integral is an anti-derivative.

A weird quirk of definite integrals as area is that sometimes the area goes negative! This happens whenever the function drops below the x-axis.



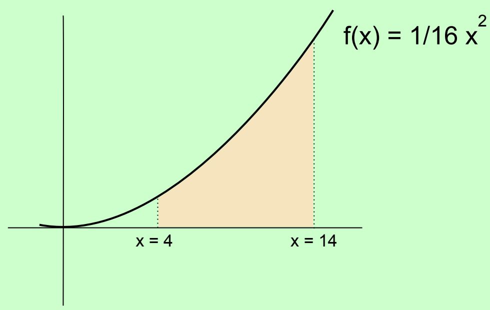
Here are some examples.

Example

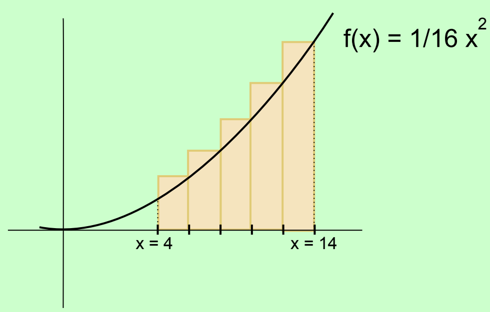
Riemann Sum I

Problem Approximate $\int_4^{14} \frac{1}{16}x^2dx$ using five rectangles.

First, if we graph this function, we see it looks something like (not to scale)



We will approximate the area in this case with five rectangles:

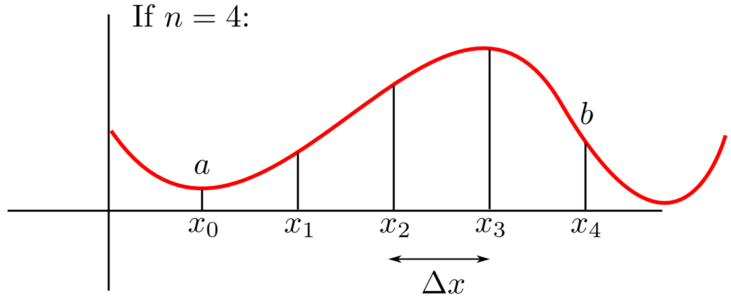


This is called the "right rectangle rule", since it is the top right of the rectangles that match the height of the function.

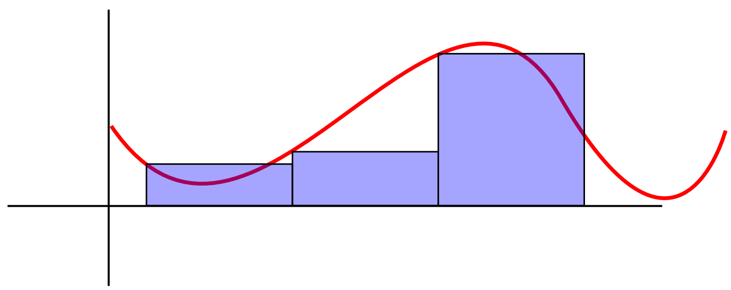
We just need to find the areas of these rectangles, add them up, and be done. We see the width of each rectangle is 2, since the distance from a to b is 10, and there are 5 rectangles. The heights can be found by plugging in $x = 6, 8, 10, 12, 14$ into the function. We see the heights are $36/16 = 2.25$, $64/16 = 4$, $100/16 = 6.25$, $144/16 = 9$, and $196/16 = 12.25$. Adding these up, we see the total area is $2.25 + 4 + 6.25 + 9 + 12.25 = 33.75$.

LIST OF NUMERIC INTEGRATION FORMULAS

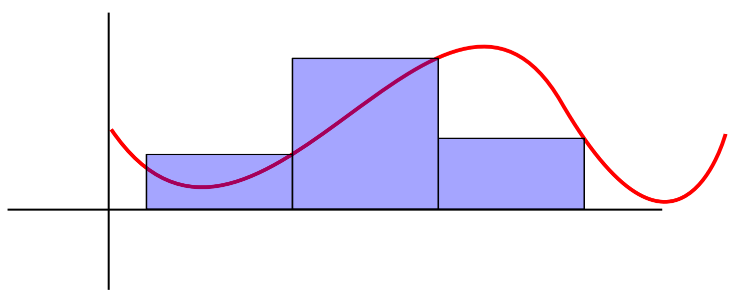
We can write a formula for approximating with n rectangles or other shapes. Let Δx be the width of the shapes, and $x_0 = a$, x_1, x_2, x_3, $\ldots$, x_{n-1}, $x_n = b$ be the values along the x axis. It looks like the following for $n = 4$:



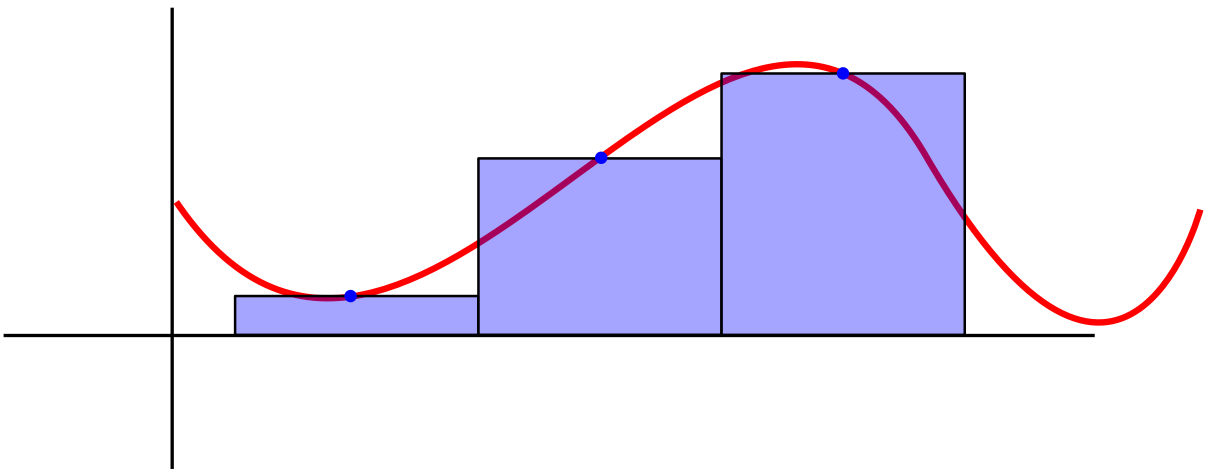
What follows are various methods for approximating the area. First, there are three rectangle-based approximations:



n left rectangle approximation = $\Delta x(f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}))$

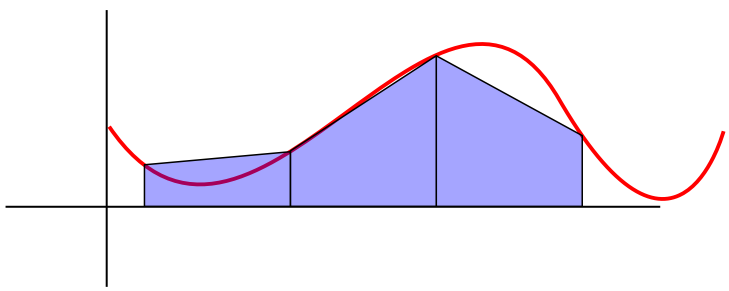


n right rectangle approximation = $\Delta x(f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n))$



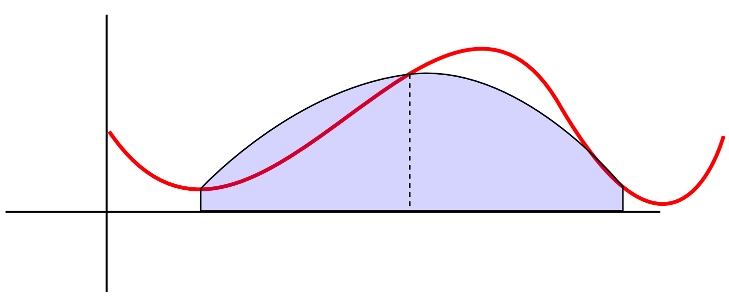
n midpoint rectangle approximation = $\Delta x \left(f\left(\frac{x_0+x_1}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right)$

If we instead use trapezoids to approximate the area, which is more accurate, we get this formula



n trapezoid approximation = $\frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n))$

Finally, if n is even, then we can approximate with quadratic curves, which is more accurate yet. We have



Simpson's rule =
$\frac{\Delta x}{3}(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n))$

Here is an example.

Example

Numeric integration techniques example

Problem Use left rectangles, trapezoids, and Simpson's rule to approximate $\int_0^8 \frac{1}{x+1}$. In each case use $n = 4$.

If $n = 4$, then we will need to know $f(x_0)$, $f(x_1)$, $f(x_2)$, $f(x_3)$, and $f(x_4)$. The five points $x_0, \ldots, x_4$ are evenly spaced on the interval from 0 to 8. To find the spacing, we take $\Delta x = \frac{b-a}{n} = \frac{8-0}{4} = 2$. This makes $\Delta x = 2$, and hence $x_0 = 0$, $x_1 = 2$, $x_2 = 4$, $x_3 = 6$, and $x_4 = 8$. We can then compute each f value. For example, $f(x_3) = f(6) = \frac{1}{6+1} = \frac{1}{7}$. The other f values are

$$f(x_0) = 1, \quad f(x_1) = \tfrac{1}{3}, \quad f(x_2) = \tfrac{1}{5}, \quad f(x_3) = \tfrac{1}{7}, \quad f(x_4) = \tfrac{1}{9}.$$

Now we just have to use the various formulas. Using rectangles, we have

$$\begin{aligned}\int_0^8 \frac{1}{x+1} &\approx \Delta x(f(x_0) + f(x_1) + f(x_2) + f(x_3)) \\ &= 2\left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}\right) \\ &= 2\left(\frac{176}{105}\right) \\ &= \frac{352}{105} \approx \boxed{3.35}\end{aligned}$$

By the way, please use a calculator to help with these calculations — they are very tedious to do by hand!

Using trapezoids, we have

$$\begin{aligned}\int_0^8 \frac{1}{x+1} &\approx \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)) \\ &= \frac{2}{2}\left(1 + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{1}{9}\right) \\ &= 1\left(\frac{776}{315}\right) \\ &= \frac{776}{315} \approx \boxed{2.46}\end{aligned}$$

Using Simpson's rule (which approximates with parabolas),

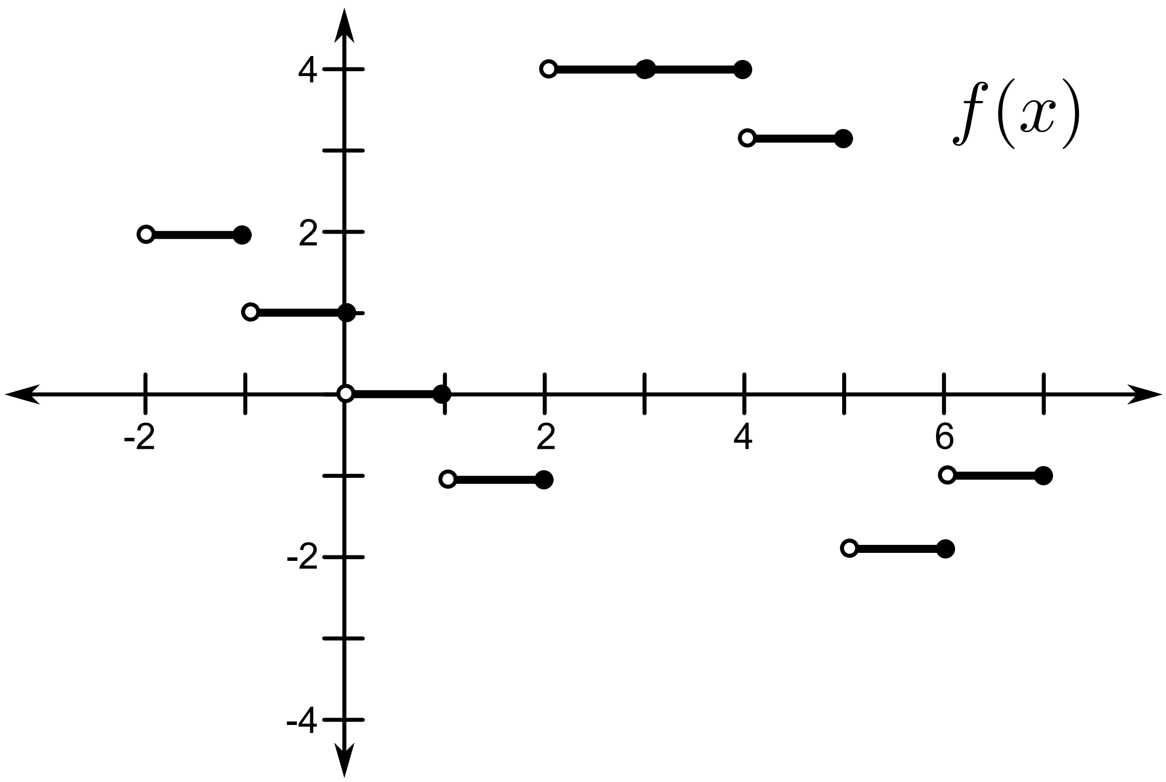
$$\begin{aligned}\int_0^8 \frac{1}{x+1} &\approx \frac{\Delta x}{3}(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4))\\ &= \frac{2}{3}\left(1+\frac{4}{3}+\frac{2}{5}+\frac{4}{7}+\frac{1}{9}\right)\\ &= \frac{2}{3}\left(\frac{1076}{315}\right)\\ &= \frac{2152}{945} \approx \boxed{2.28}\end{aligned}$$

(By the way, the exact value is $\ln(9) \approx 2.20$)

Chapter 57

HOMEWORK: NUMERIC INTEGRATION TECHNIQUES

1. Given the picture of $f(x)$, find the following definite integrals.



a. $\int_{-2}^{2} f(x)dx$.
2
ans

b. $\int_{2}^{7} f(x)dx$.
8
ans

c. $\int_{0}^{6} f(x)dx$.
8
ans

d. $\int_{4}^{7} f(x)dx$.
0
ans

e. $\int_{-2}^{7} f(x)dx$.
10
ans

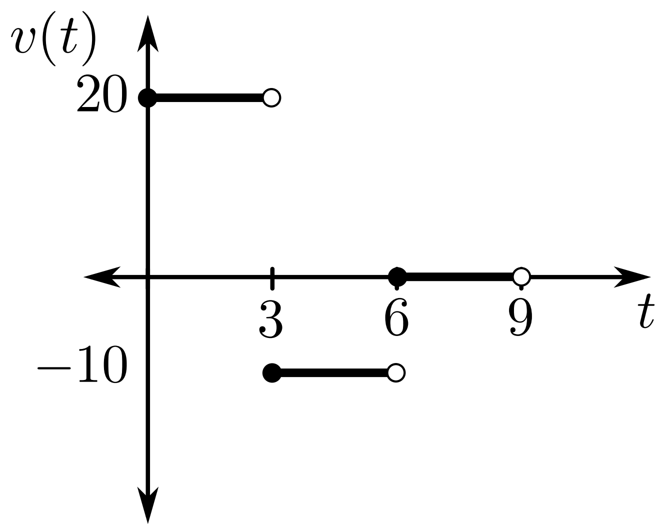
f. $\int_{3}^{3} f(x)dx$ (what do you think this even means?)
0 — You can think of this as an infinitely thin rectangle.
ans

g. $\int_{-1}^{0} f(x)dx$.
1
ans

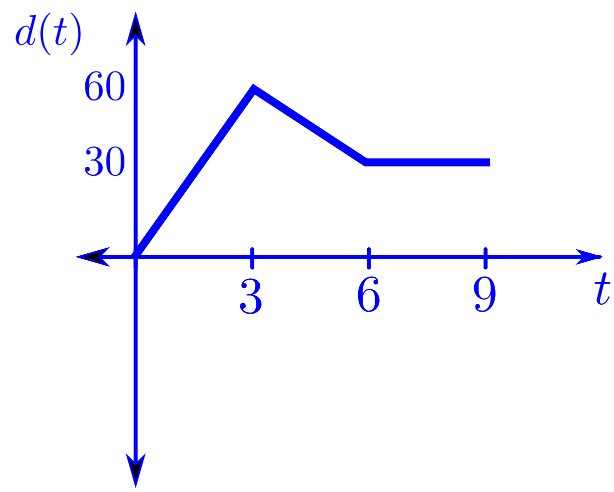
h. $\int_{0}^{1.5} f(x)dx$.
$-1/2$
ans

i. $\int_{-1.5}^{4.5} f(x)dx$
10.5
ans

2. Remember those problems from homework 1 where we started with a velocity graph and then drew the position graph? Let's try that again.



a. Draw the position graph corresponding to this velocity graph next to the graph above.



ans

b. Find the following three integrals based on the original graph $v(t)$:

$$\int_0^3 v(t)dt, \quad \int_0^6 v(t)dt, \quad \int_0^9 v(t)dt.$$

60, 30, 30

ans

c. Your answers for parts (a) and (b) should look the same in some way. How do they look the same, and why did it work out this way?

The answers are the key y-values from the graph in part (a). This is the same, since in both cases you are doing the same thing: takeing $\Delta x \cdot y$ (or $\Delta t \cdot$ velocity and adding it up as you go.

ans

3. Using 6 rectangles and using the left rectangle rule, estimate the area under the curve of the following functions from $x = 0$ to $x = 3$.

   a. $e(x) = 2 - x$.
   ≈ 2.25
   ans

   b. $f(x) = -\frac{1}{9}x^2 + 1$
   ≈ 2.24
   ans

   c. $g(x) = e^x$
   ≈ 14.71
   ans

4. For $e(x)$ in problem 3, find the exact area using the formula for the area of a triangle.
   1.5
   ans

5. For $f(x)$ in problem 3:

   a. If you haven't done so already, sketch a picture of the graph as well as the rectangles you used to approximate the area.

b. Is your approximation from problem 3 an *over-estimate* or an *under-estimate*? How do you know?
Overestimate, since it looks like the rectangles cover too much area.
ans

6. Watch the Khan Academy video on the Trapezoidal method of finding the area under the curve.
7. Using 6 trapezoids that match the height of the graph, estimate the area under the curve of the following functions from $x = 0$ to $x = 3$.

   a. $e(x) = 2 - x$.
   1.5
   ans
   b. $f(x) = -\frac{1}{9}x^2 + 1$
   1.99
   ans
   c. $g(x) = e^x$
   19.48
   ans

8. Watch this Khan Academy like video on the midpoint formula: click here
9. Using 6 rectangles, use the midpoint rule to approximate the area under the curve from $x = 0$ to $x = 3$.

   a. $e(x) = 2 - x$.
   1.5
   ans
   b. $f(x) = -\frac{1}{9}x^2 + 1$
   2.03
   ans
   c. $g(x) = e^x$
   18.31
   ans

10. I couldn't find a Khan Academy explanation of Simpson's rule, so here is another video by Patrick at Just Math Tutoring.
11. Use Simpson's rule to approximate the area under the curve. Use 6 intervals from $x = 0$ to $x = 3$.

   a. $e(x) = 2 - x$.
   1.5
   ans
   b. $f(x) = -\frac{1}{9}x^2 + 1$
   2
   ans
   c. $g(x) = e^x$
   ≈ 19.09
   ans

12. Note that the actual answers for the area under the curve from $x = 0$ to $x = 3$ are

    a. 1.5 for $e(x) = 2 - x$.
    b. 2 for $f(x) = -\frac{1}{9}x^2 + 1$
    c. 19.086 for $g(x) = e^x$.

    Given these answers, rate the following rules from most accurate to least accurate based on the answers from this homework: left rectangle rule, trapezoid rule, midpoint rule, Simpson's rule.

Chapter 58

FUNDAMENTAL THEOREM OF CALCULUS

In the previous two sections, we saw that the area under a curve can be found using more and more rectangles. However, this process can be tedious and not very enlightening. There is a powerful theorem that allows us to compute area under the curve quickly in many cases.

Fundamental Theorem of Calculus

Given a function $f(x)$ where $F(x)$ is an anti-derivative of $f(x)$, we have

$$\int_a^b f(x)dx = F(b) - F(a)$$

This is a really remarkable theorem. At first blush, finding the area under a curve and finding the slope of a tangent line have nothing in common. What this theorem is saying is that these are intimately tied, and in fact that are exactly inverse operations. That is, one undoes the other. Moreover, this gives an exact answer to integral problems, something that eluded us in the previous sections.

For example, let's say we wanted to solve the following:

Problem Find $\int_2^5 x^2 dx$.

To use the fundamental theorem, we need an anti-derivative of $f(x) = x^2$. The anti-derivative in this case, by the inverse power rule, is $x^3/3$. Therefore $F(x) = x^3/3$. The answer then is $F(b) - F(a)$, which is

$$\begin{aligned}\int_2^5 x^2 dx &= \frac{5^3}{3} - \frac{2^3}{3} \\ &= \frac{125}{3} - \frac{8}{3} \\ &= \frac{117}{3} \\ &= \boxed{39}\end{aligned}$$

The area under the curve is 39. Note that $F(b) - F(a)$ is sometimes denoted $F(x)\Big|_a^b$. We'll see that in the following examples.

Example

Fundamental Theorem Examples

- Problem Find $\int_{-2}^{1} 3x^2 dx$.

To solve this, we find the anti-derivative of $3x^2$, and then plug in the end points and subtract the result. First note that $\int 3x^2 dx = x^3 + C$.

$$\begin{aligned}
\int_{-2}^{1} 3x^2 dx &= x^3 + C\Big|_{-2}^{1} \\
&= (1^3 + C) - ((-2)^3 + C) \\
&= 1 + C - (-8 + C) \\
&= 1 + C + 8 - C \\
&= \boxed{9}
\end{aligned}$$

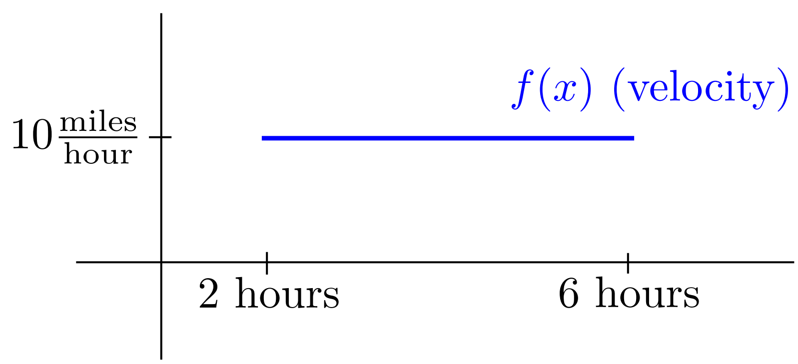
Notice how the constants of integration cancel? For definite integral problems, we can essentially ignore the constant of integration for this reason.

- Problem Find $\int_0^6 \frac{1}{2}x^2 + \frac{1}{3}x dx$.

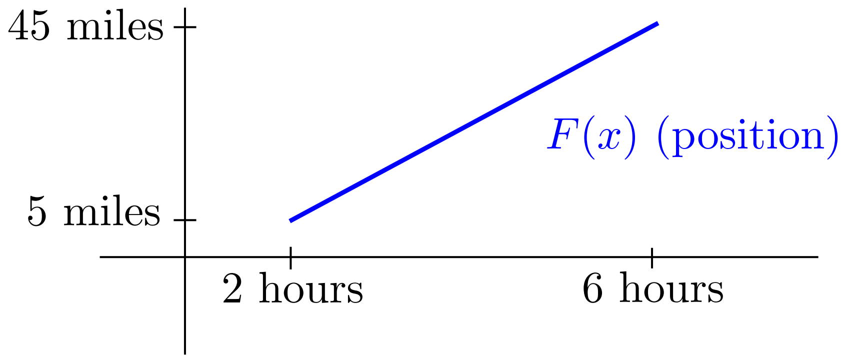
We integrate and then evaluate again.

$$\begin{aligned}
\int_0^6 \frac{1}{2}x^2 + \frac{1}{3}x dx &= \frac{1}{2}\int_0^6 x^2 + \frac{1}{3}\int_0^6 x dx \\
&= \left(\frac{1}{2}\frac{x^3}{3} + \frac{1}{3}\frac{x^2}{2}\right)\Big|_0^6 \\
&= \left(\frac{x^3}{6} + \frac{x^2}{6}\right)\Big|_0^6 \\
&= \frac{x^3 + x^2}{6}\Big|_0^6 \\
&= \left(\frac{6^3 + 6^2}{6}\right) - \left(\frac{0^3 + 0^2}{6}\right) \\
&= \frac{252}{6} - \frac{0}{6} \\
&= \boxed{42}.
\end{aligned}$$

Why does the fundamental theorem of calculus work? As we have seen earlier, it is sometimes easiest to see in terms of position versus velocity. Let $F(x)$ be the position function of a car, let $f(x)$ be the velocity function of the car, and let x be the time in hours. For simplicity, let's say $f(x)$ is constant from $x = 2$ to $x = 6$:



So if this is the velocity function, what is happening to the position function? As we've seen, we just need to **multiply**: its the $10\frac{\text{miles}}{\text{hour}}$ times the four hours, gives a change of 40 miles. That means $F(x)$ looks something like this:



Note it need not start at 5 miles, but that is one possibility.

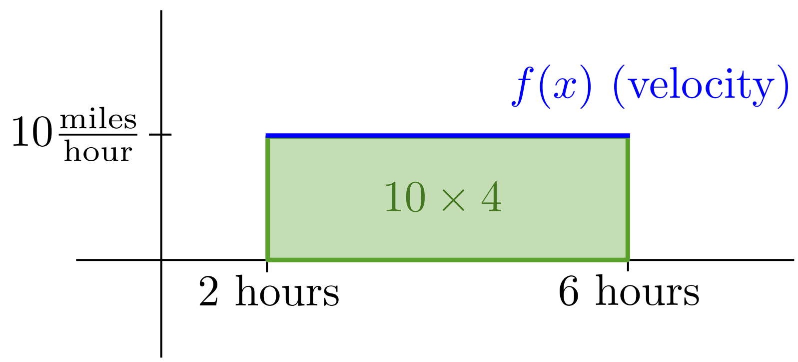
Now let's look at the fundamental theorem again:

Fundamental Theorem of Calculus

Given a function $f(x)$ where $F(x)$ is an anti-derivative of $f(x)$, we have

$$\int_a^b f(x)dx = F(b) - F(a)$$

Can you see why the fundamental theorem worked out in this case? We see see $F(b) - F(a)$ is just the change of position between hour 2 and hour 6, which is $45 - 5$ or 40 miles. What is $\int_a^b f(x)dx$? Well, that's the area under the velocity curve. How do you find area? That's right, it's **multiplication**! In particular, it is $10\frac{\text{miles}}{\text{hour}}$ times the four hours, again giving a change of 40 miles.



You can see how both area under the curve and antiderivatives come down to the same basic calculation. That's why the fundamental theorem of calculus can claim they are the same thing.

If the velocity function is more complicated, this still works. We can think of a more complicated function as a combination of these constant functions.

Chapter 59

HOMEWORK: THE FUNDAMENTAL THEOREM OF CALCULUS

1. What is the fundamental theorem of calculus saying? Think intuitively, big picture, etc. Use your own words.
   There are some possible answers I'm looking for:

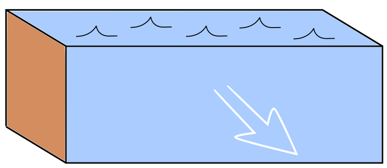
   - Area under the curve can be computed using an anti-derivative
   - Integrals and derivatives cancel!
   - Finding area is related to finding slopes
   - Finding area and finding slope are inverse operations
   - To find how far something has gone using a velocity graph, use the area under the curve.

   ans

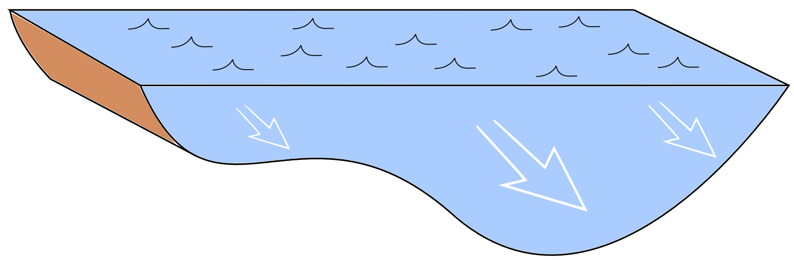
Chapter 60

PROJECT: MEASURING STREAMFLOW

Purpose of the project: Apply numeric integration techniques to a real-world problem.



$$\begin{aligned}\text{flow} &= \text{area} \cdot \text{velocity}\\ &= \text{width} \cdot \text{depth} \cdot \text{velocity}\end{aligned}$$



$$\text{flow} = ??$$

How do you measure streamflow? The basic idea is simple: it is area times velocity. For example, suppose you had a river that was 20 feet wide, 3 feet deep, and had water moving at 2 feet per second. Then we multiply the 20 and the 3 to get an area of 60 ft^2, and the multiply by the 2 ft per second to get 120 ft^3 per second. This works great if you have a rectangular river where the water moves at a constant velocity. But what if the river is not a rectangle? What if the velocity changes depending on where in the river you are? How can you find the streamflow? And what does this have to do with numeric integration techniques?

1. Come up with two different ways to find streamflow in a river. Why do they work? And how do they relate to the numeric integration techniques we studied?
2. Try both methods on the virtual river (see river.html). I recommend using a spreadsheet to record your data and to automatically do the calculations for you.
3. If you have access to a flow meter or some method of calculating water velocity, try out both methods in a real river. Which method do you feel worked better? Why?

Chapter 61

PROJECT: QUAKE LAKE

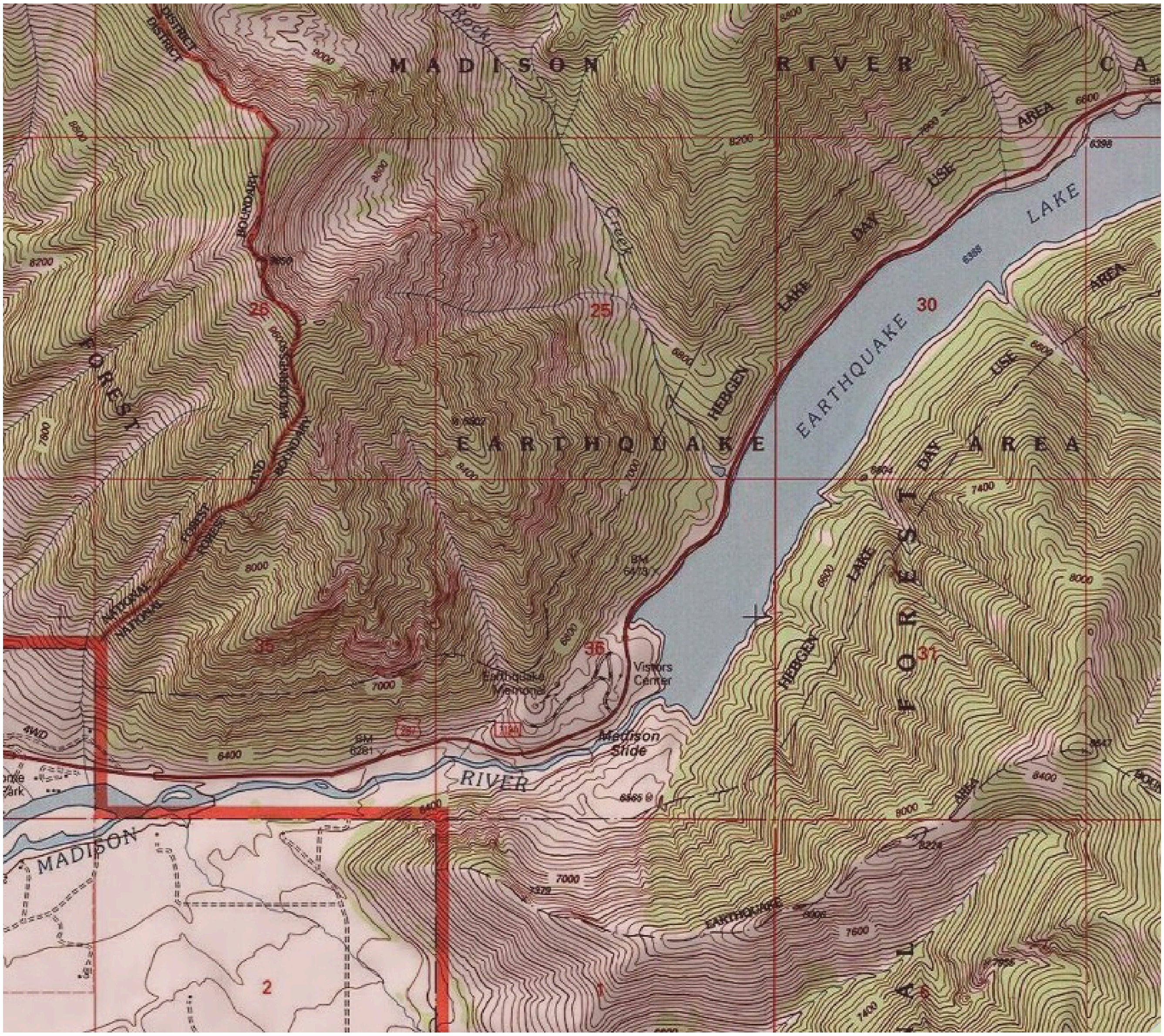
Purpose of the project: Apply numeric integration techniques to a real-world problem.

(source: library.usgs.gov, photo by J.B. Hadley)

In 1959, an 7.2 magnitude earthquake caused a massive landslide in southwest Montana near Yellowstone National Park. 28 people were killed in the quake, and the landslide created a natural dam of the Madison river which created a new lake called Earthquake Lake or Quake Lake. See this travel blog for some additional photos and diagrams. The area where the land slid away is still mostly barren of trees today.

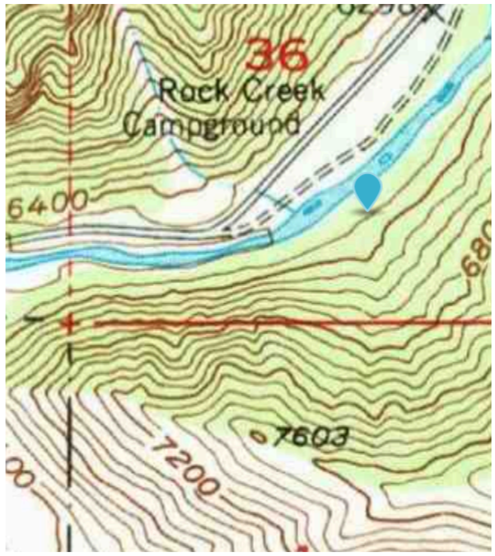
The main question behind this project: roughly how much volume of rock and earth was moved in the landslide?

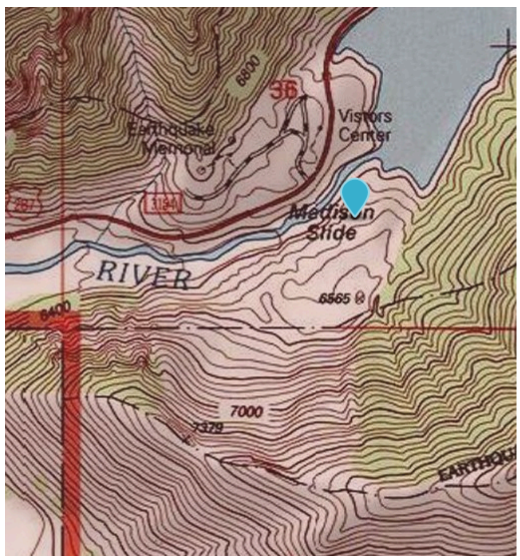
1. What are some possible ways to figure how much volume?
2. Using Google Earth and USGS topographical maps, give a very rough estimate of how many cubic meters or cubic feet of earth was moved in the landslide.
3. Find an online reference for the volume of the landslide. Is your number high, low, or just right? Can you guess why your estimate was off in the direction it was?



(source: https://ngmdb.usgs.gov/topoview/)

Above is a topographical map of the area today. Below is two pictures zoomed in of the same area, right where the rock slide occured. The left image is an older map from before the rock slide, and the second is the same newer map image from after the rock slide.





(source: https://ngmdb.usgs.gov/topoview/)

PART VII

RULES FOR INTEGRATION

CHAPTER 62

POWER, EXPONENTIAL, TRIG, AND LOGARITHM RULES

We've already seen the inverse power rule, but here it is again:

$$\boxed{\int x^m dx = \frac{x^{m+1}}{m+1} + C}$$

Note that this only works if $m \neq -1$. However, we haven't seen how this works with fractional and negative powers yet. We'll do some examples of this.

We can also "undo" the derivatives for exponential, logarithmic functions, or trigonometric functions.

$$\boxed{\int e^x dx = e^x + C}$$

$$\boxed{\int \frac{1}{x} dx = \ln(x) + C}$$

$$\boxed{\int \sin(x) dx = -\cos(x) + C}$$

$$\boxed{\int \cos(x) dx = \sin(x) + C}$$

On to the examples. Recall that $x^{-m} = \frac{1}{x^m}$ and $x^{1/m} = \sqrt[m]{x}$.

Example

Power rule with fractional and negative powers

- Problem Find $\int_4^9 x^{-3} + 2x^{1/2} dx$.

Negative and fractional powers work the same way as the positive powers we've been working with. We see

$$\begin{aligned}
\int_4^9 x^{-3} + 2x^{1/2} dx &= \int_4^9 x^{-3} dx + 2\int_4^9 x^{1/2} dx \\
&= \left(\frac{x^{-2}}{-2}\right) + 2\left(\frac{x^{3/2}}{3/2}\right)\Big|_4^9 \\
&= \frac{1}{-2x^2} + 2\frac{2x^{3/2}}{3}\Big|_4^9 \\
&= \frac{1}{-2x^2} + \frac{4x^{3/2}}{3}\Big|_4^9 \\
&= \left(\frac{1}{-2(9)^2} + \frac{4(9)^{3/2}}{3}\right) - \left(\frac{1}{-2(4)^2} + \frac{4(4)^{3/2}}{3}\right) \\
&= \left(-\frac{1}{162} + \frac{108}{3}\right) - \left(-\frac{1}{32} + \frac{32}{3}\right)
\end{aligned}$$

At this point, let's just type it into a calculator to get an approximate answer. We see that the integral is about $\boxed{23.36}$.

- Problem Find $\int_1^2 \frac{1}{x^4} dx$.

This problem becomes an inverse power rule problem if we notice that $\frac{1}{x^4} = x^{-4}$. We see

$$
\begin{aligned}
\int_1^2 \frac{1}{x^4}dx &= \int_1^2 x^{-4}dx \\
&= \frac{x^{-5}}{-5}\bigg|_1^2 \\
&= -\frac{1}{5x^5}\bigg|_1^2 \\
&= \left(-\frac{1}{5(2)^5}\right) - \left(-\frac{1}{5(1)^5}\right) \\
&= -\frac{1}{160} + \frac{1}{5} \\
&= -\frac{1}{160} + \frac{32}{160} \\
&= \boxed{\frac{31}{160}}.
\end{aligned}
$$

- Problem Find $\int_{25}^{100} \sqrt{x}dx$.

If we remember that $x^{1/2} = \sqrt{x}$, this is a power rule problem.

$$\begin{aligned}
\int_{25}^{100} \sqrt{x}dx &= \int_{25}^{100} x^{1/2}dx \\
&= \frac{x^{3/2}}{3/2}\Big|_{25}^{100} \\
&= \frac{2x^{3/2}}{3}\Big|_{25}^{100} \\
&= \left(\frac{2(100)^{3/2}}{3}\right) - \left(\frac{2(25)^{3/2}}{3}\right) \\
&= \frac{2000}{3} - \frac{250}{3} \\
&= \boxed{\frac{1750}{3}}.
\end{aligned}$$

Now for logarithmic and exponential functions.

Example

Logarithmic and Exponential functions and integration

- Problem Find $\int_1^5 \frac{2}{x}dx$.

For this problem, you may be tempted to write this as $2x^{-1}$ and use the inverse power rule. Good instincts, but in this case it won't work (try it: it leads to division by zero!). So instead, let's invert the natural logarithm derivative.

$$\begin{aligned}\int_1^5 \frac{2}{x}dx &= 2\int_1^5 \frac{1}{x}dx\\ &= 2\left(\ln(x)\right)\Big|_1^5\\ &= (2\ln(5)) - (2\ln(1))\\ &= 2\ln(5) - 0\\ &= 2\ln(5) \approx \boxed{3.2}.\end{aligned}$$

- Problem Find $\int_{-2}^{2} 7e^x dx$.

e^x is the best function for calculus. Doesn't change with integration!

$$\begin{aligned}\int_{-2}^{2} 7e^x dx &= 7\int_{-2}^{2} e^x dx\\ &= 7(e^x)\Big|_{-2}^{2}\\ &= (7e^2) - (7e^{-2})\end{aligned}$$

Not a lot to do to simplify this, but we can get a decimal approximation: $\approx \boxed{50.78}$.

Chapter 63

HOMEWORK: POWER, EXPONENTIAL, TRIG, AND LOGARITHMIC RULES

1. Compute the following definite integrals.

    a. $\int_2^3 x^3 + 2\sqrt{x}dx$
    19.04
    ans

    b. $\int_{-2}^3 (x+5)^2 dx$
    ≈ 161.7
    ans

    c. $\int_0^1 e^x dx$
    $e - 1 \approx 1.718$
    ans

    d. $\int_{-1}^1 3e^x dx$
    7.05
    ans

    e. $\int_1^e \frac{3}{x} + \frac{x}{3}\ dx$
    ≈ 4.06
    ans

2. Approximate $\int_0^1 x^2 dx$ using 4 rectangles. Then find $\int_0^1 x^2$ exactly using an anti-derivative. How far off is the approximation?
Approximation ≈ 0.22, the actual is $\frac{1}{3} \approx .33$, so the difference is about 0.11 or 50 error which isn't great. As we know, the rectangles don't always do such a good job.
ans

Chapter 64

U-SUBSTITUTION

Recall the chain rule:

$$\boxed{\frac{d}{dx}f(g) = f'(g) \cdot g'}$$

The method of "u-substitution" is a way of doing integral problems that undo the chain rule. It also helps deal with constants that crop up.

u-substitution:

1. Identify an "inside" function whose derivative is multiplied on the outside, possibly with a different constant. Call this "inside" function u.
2. Compute $\frac{du}{dx}$ and solve for dx.
3. Use substitution to replace $x \to u$ and $dx \to du$, and cancel any remaining x terms if possible.
4. Integrate with respect to u. If at this point you still have any xs in your problem, either you made a mistake or the method of u-substitution will not work for this problem.
5. Substitute back the xs back into the answer before evaluating the definite integral.

Let's do some examples.

Example u-**substitution**

Problem Find $\int_0^3 e^{(-5x)}dx$.

We will follow the steps of u substitution.

1. In this case, the "inside function" is $u = -5x$.
2. If we compute $\frac{du}{dx}$, we see the derivative of $-5x$ is -5. Hence $\frac{du}{dx} = -5$, and we have solving for dx

$$\begin{aligned}\frac{du}{dx} &= -5\\ du &= -5dx\\ -\frac{1}{5}du &= dx\end{aligned}$$

3. Going back to the original problem and using substitution $u = -5x$ and $dx = -\frac{1}{5}du$, we thus have:

$$\begin{aligned}\int_0^3 e^{-5x}dx \int_0^3 e^u dx\\ &= \int_0^3 e^u \left(-\frac{1}{5}du\right)\\ &= \int_0^3 -\frac{1}{5}e^u du.\end{aligned}$$

4. Integrating with respect to u,

$$\begin{aligned}\int_0^3 -\frac{1}{5}e^u du &= -\frac{1}{5}\int_0^3 e^u du\\ &= -\frac{1}{5}e^u\Big|_0^3\end{aligned}$$

5. We now substitute $u = -5x$. We have

$$\begin{aligned}
-\frac{1}{5}e^u\Big|_0^3 &= -\frac{1}{5}e^{-5x}\Big|_0^3 \\
&= \left(-\frac{1}{5}e^{-15}\right) - \left(-\frac{1}{5}e^{-5(0)}\right) \\
&= -\frac{e^{-15}}{5} + \frac{1}{5} \\
&= \frac{1-e^{-15}}{5} \\
&\approx \boxed{0.2}
\end{aligned}$$

There we go.

Example u-**substitution**

Problem Find the indefinite integral $\int x(x^2+1)^7 dx$.

Again, we will go through the steps of u-substitution.

1. The inside function in this case is x^2+1. We can see that the derivative is $2x$, and this is good since there is an x multiplied out in front (the 2 is just a constant we can deal with.) Set $u = x^2+1$.
2. We see $\frac{du}{dx} = 2x$, and hence solving for dx we have $\frac{du}{2x} = dx$.
3. Subbing in $u = x^2+1$ and $dx = \frac{du}{2x}$, we have

$$\begin{aligned}\int x(x^2+1)^7 dx &= \int xu^7 dx \\ &= \int xu^7 \left(\frac{du}{2x}\right) \\ &= \int \frac{x}{2x} u^7 du \\ &= \int \frac{1}{2} u^7 du\end{aligned}$$

Great, the xs are all gone!

4. We can now integrate with respect to u.

$$\begin{aligned}\int \frac{1}{2} u^7 du &= \frac{1}{2} \int u^7 du \\ &= \frac{1}{2} \frac{u^8}{8} \\ &= \frac{u^8}{16}\end{aligned}$$

5. Finally, we sub in the xs again using $u = x^2 + 1$.

$$\frac{u^8}{16} = \boxed{\frac{(x^2+1)^8}{16} + C}$$

Since this is an indefinite integral, we add the constant of integration.

Example u-**substitution**

Problem Find the indefinite integral $\int \frac{8(\ln(x))^3}{x} dx$.

Again, we will go through the steps of u-substitution.

1. The inside function in this case is $\ln(x)$. We can see that the derivative is $\frac{1}{x}$, and this is good since there is an x dividing the rest of the problem. Set $u = \ln(x)$.
2. We see $\frac{du}{dx} = \frac{1}{x}$, and hence solving for dx we have $\frac{du}{\frac{1}{x}} = dx$.
3. Subbing in $u = \ln(x)$ and $dx = \frac{du}{\frac{1}{x}}$, we have

$$\begin{aligned}\int \frac{8(\ln(x))^3}{x} dx &= \int \frac{8u^3}{x} \cdot \frac{du}{\frac{1}{x}} \\ &= \int \frac{8u^3}{x \cdot \frac{1}{x}} du \\ &= \int \frac{8u^3}{1} du \\ &= \int 8u^3 du\end{aligned}$$

   Great, the xs are all gone!
4. We can now integrate with respect to u.

$$\begin{aligned}\int 8u^3 du &= 8 \cdot \frac{1}{4} u^4 \\ &= 2u^4\end{aligned}$$

5. Finally, we sub in the xs again using $u = \ln(x)$.

$$\int \frac{8(\ln(x))^3}{x} dx = \boxed{2(\ln(x))^4 + C}.$$

If the inside function is linear, the u-substitution is much simpler, and there is even a formula for it (just like in the $\int e^{-5x} dx$ example above). By the chain rule with $g = mx + b$ and $g' = m$, we have

$$\begin{aligned}
\frac{d}{dx}\frac{1}{m}f(mx+b) &= \frac{1}{m}\frac{d}{dx}f(mx+b) \\
&= \frac{1}{m}\left(f'(g)\cdot g'\right) \\
&= \frac{1}{m}f'(mx+b)\cdot m \\
&= \frac{m}{m}f'(mx+b) \\
&= f'(mx+b)
\end{aligned}$$

If we integrate both sides of this equation, we have the following useful rule which I call the "chain rule shortcut":

$$\boxed{\int f'(mx+b)dx = \frac{1}{m}f(mx+b)}$$

This is especially important if $f(x) = e^{mx}$. In this case, we have

$$\boxed{\int e^{mx}dx = \frac{1}{m}e^{mx}}$$

In other words, you integrate just like normal without any u-substitution, and then add a $\frac{1}{m}$ factor for the fact that you have $mx+b$ inside the function instead of just an x.

Let's see a couple of examples of this:

Example **Chain rule shortcut**

- Problem Find $\int(6x-3)^4dx$.

Using the inverse power rule, we see this becomes $\frac{(6x-3)^5}{5}$. However, we need that $\frac{1}{m}$ factor since the problem has a $6x-3$ in it. So the final answer is

$$\frac{1}{6}\frac{(6x-3)^5}{5} = \frac{(6x-3)^5}{30} + C.$$

You'd get the same thing doing a full u-substitution with $u = 6x-3$. This way, though, you save some time by just multiplying by $\frac{1}{6}$.

- Problem Find $\int_{-2}^{-1}(3x+5)^7dx$.

Again, we use the u-sub shortcut — we just need to do power rule and remember a $\frac{1}{m}$ factor, which in this case is $\frac{1}{3}$.

$$\begin{aligned}\int_{-2}^{-1}(3x+5)^7dx &= \frac{1}{3}\frac{(3x+5)^8}{8}\Big|_{-2}^{-1} \\ &= \frac{(3x+5)^8}{24}\Big|_{-2}^{-1} \\ &= \left(\frac{(3(-1)+5)^8}{24}\right)-\left(\frac{(3(-2)+5)^8}{24}\right) \\ &= \frac{256}{24}-\frac{1}{24} \\ &= \frac{255}{24} \\ &= \frac{83}{8}\end{aligned}$$

- Problem Find $\int e^{0.05t}dt$.

The antiderivative here is just $\frac{1}{0.05}e^{0.05t}$. We can plug $1/0.05$ into a calculator and get 20 , so the answer is $\boxed{20e^{0.05t}+C}$.

Chapter 65

HOMEWORK: U-SUBSTITUTION

1. As a review of the chain rule from derivatives, find $\frac{d}{dx}e^{-x^2+1}$.
   $2xe^{x^2+1}$
   ans
2. Read through section 6A again and then read through section 6B.
3. Watch the Khan Academy video u-substitution.
4. Compute the following indefinite integral using the method of u-substitution.

$$\int(5x^4+2x)e^{x^5+x^2}dx$$

   $e^{x^5+x^2}+C$
   ans
5. Watch another example of u-substitution: u-substitution 2.
6. Compute the following indefinite integral using the method of u-substitution.

$$\int\frac{2x}{x^2-5}dx$$

   $\ln(x^2-5)+C$
   ans
7. Reread the part about the chain rule shortcut for u-substitution in chapter 6 of the online notes, and reread Example 6B.2. Then try the following problems.

   a. $\int e^{-3x}dx.$
   $\frac{1}{-3}e^{-3x}+C$
   ans
   b. $\int\left(\frac{1}{2}x-1\right)^4dx$
   $.4\left(\frac{1}{2}x-1\right)^5+C$
   ans

   .
   c. $\int_0^2(5x-3)^3dx.$
   116
   ans
   d. $\int\frac{1}{7x-2}dx$
   $\frac{1}{7}\ln(7x-2)+C$
   ans
   e. $\int\frac{1}{\sqrt{0.5x-1}}dx$

$4\sqrt{0.5x-1}+C$
ans

8. Try some more u-substitution integrals.

   a. $\int(8x^3)(x^4+1)^2 dx$

      1. We see that $u=x^4+1$ in this case.
      2. We see $\frac{du}{dx}=4x^3$, so $dx=\frac{1}{4x^3}du$.
      3. We have

$$
\begin{aligned}
&= \int (8x^3)u^2\frac{1}{4x^3}du \\
&= \int 8u^2\frac{1}{4}du \\
&= 2\int u^2 du
\end{aligned}
$$

      4. Integrating we have

$$2\int u^2 du = 2\left(\tfrac{1}{3}u^3\right)+C=\tfrac{2}{3}u^3+C$$

      5. Substitution of $u=x^4+1$ yields

$$\tfrac{2}{3}(x^4+1)^3+C$$

   ans

   b. $\int(x^4+1)^2(8x^3)dx$
   This is exactly the same as the previous problem, just written a different way. No need to redo work.
   ans

   c. $\int(3x^2-1)e^{x^3-x}dx$

      1. We see that $u=x^3-x$ in this case.
      2. We see $\frac{du}{dx}=3x^2-1$, so $dx=\frac{1}{3x^2-1}du$.
      3. We have

$$
\begin{aligned}
&= \int (3x^2-1)e^u\frac{1}{3x^2-1}du \\
&= \int e^u du
\end{aligned}
$$

      4. Integrating we have

$$\int e^u du = e^u + C$$

5. Substitution of $u = x^3 - x$ yields

$$e^{x^3-x} + C$$

ans

d. $\int (x^2 - \frac{1}{3}) e^{x^3-x} dx$
$\frac{1}{3} e^{x^3-x} + C$
ans

e. $\int (e^x + x)^5 (e^x + 1) dx$
$\frac{1}{6} (e^x + x)^6 + C$
ans

f. $\int \frac{2x}{x^2-1} dx$
$\ln(x^2 - 1) + C$
ans

g. $\int e^x \sqrt{e^x + 1} dx$
$\frac{2}{3} (e^x + 1)^{3/2} + C$
ans

h. $\int \frac{\sin(x)}{\cos(x)} dx$
$-\ln(\cos(x)) + C$
ans

i. $\int \frac{\sqrt{\ln(x)}}{x} dx$
$\frac{2}{3} (\ln(x))^{3/2} + C$
ans

j. $\int \frac{x}{(x^2-5)^3} dx$

1. We see that $u = x^2 - 5$ in this case.
2. We see $\frac{du}{dx} = 2x$, so $dx = \frac{1}{2x} du$.
3. We have

$$= \int \frac{x}{u^3} \frac{1}{2x} du$$
$$= \int \frac{1}{2u^3} du$$
$$= \int \frac{1}{2} u^{-3} du$$

4. Integrating we have

$$\int \frac{1}{2} u^{-3} du = -\frac{1}{4} u^{-2} + C$$

5. Substitution of $u = x^2 - 5$ yields

$$-\frac{1}{4}(x^2-5)^{-2}+C$$

ans

k. $\int \frac{\cos(x)}{\sqrt{\sin(x)+1}}dx$

$2\sqrt{\sin(x)+1}+C$

ans

l. $\int \frac{1}{x(2\ln(x)+1)^4}dx.$

$-\frac{1}{6(2\ln(x)+1)^3}+C$

ans

m. $\int (e^{5x}+1)^9(e^{5x})dx.$

$\frac{1}{2}(e^{5x}+1)^{10}+C$

ans

CHAPTER 66

INTEGRAL APPLICATIONS

In derivative story problems, often you're given a function that describes the amount of something, and you're asked to find the rate of change. With integral story problems, the reverse is often true. You'll be given a function the describes the rate of change over time, and you'll be asked to get the amount.

> Problem At time $t = 0$ seconds, train starts with velocity 0 meters per second, and eventually gets up to a speed of 40 meters per second at time $t = 800$ seconds. How far has it gone from $t = 0$ to $t = 800$?

Discussion: We know that distance is equal to velocity multiplied by time. So if we can figure out the train's velocity, we can just multiply by 800 seconds and get an answer. So here are some potential answers:

Speed of the train (m/s)	Distance traveled (m)
0	0
10	8000
20	16000
30	24000
40	32000

Why can't we just go with 40 meters per second? Because the train "eventually" gets up to 40 meters per second doesn't mean it was going 40 meters per second the whole time — most of the trip it may have been going slower. So the answer of $32,000$ meters (or 32 kilometers) is an upper bound, but isn't the final answer. We need more information.

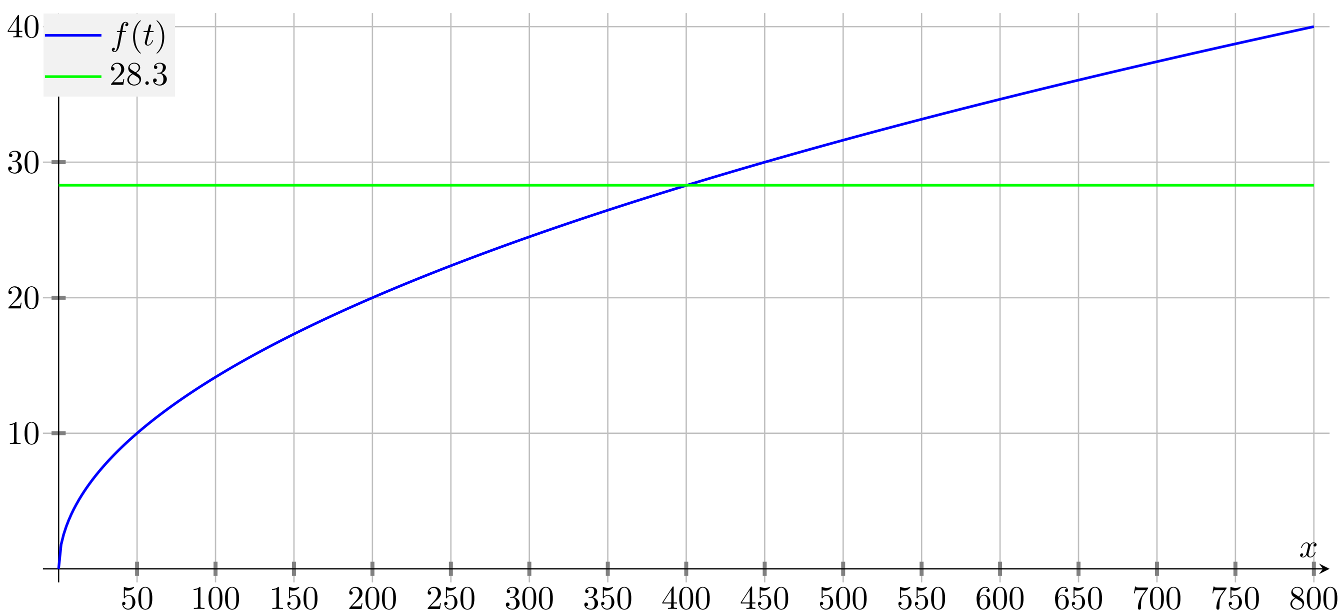
> Problem (Additional information) The speed of the train is given by the function $f(t) = \sqrt{2t}$ meters per second.

What this additional information is telling us is that there isn't just a single velocity we can use and multiply by the 800 seconds — instead, the velocity is changing all the time so there are lots of velocities. Since the train goes from 0 to 40 meters per second, can we just use the "average" which is 20 meters per second? Well, let's look at the speeds over time:

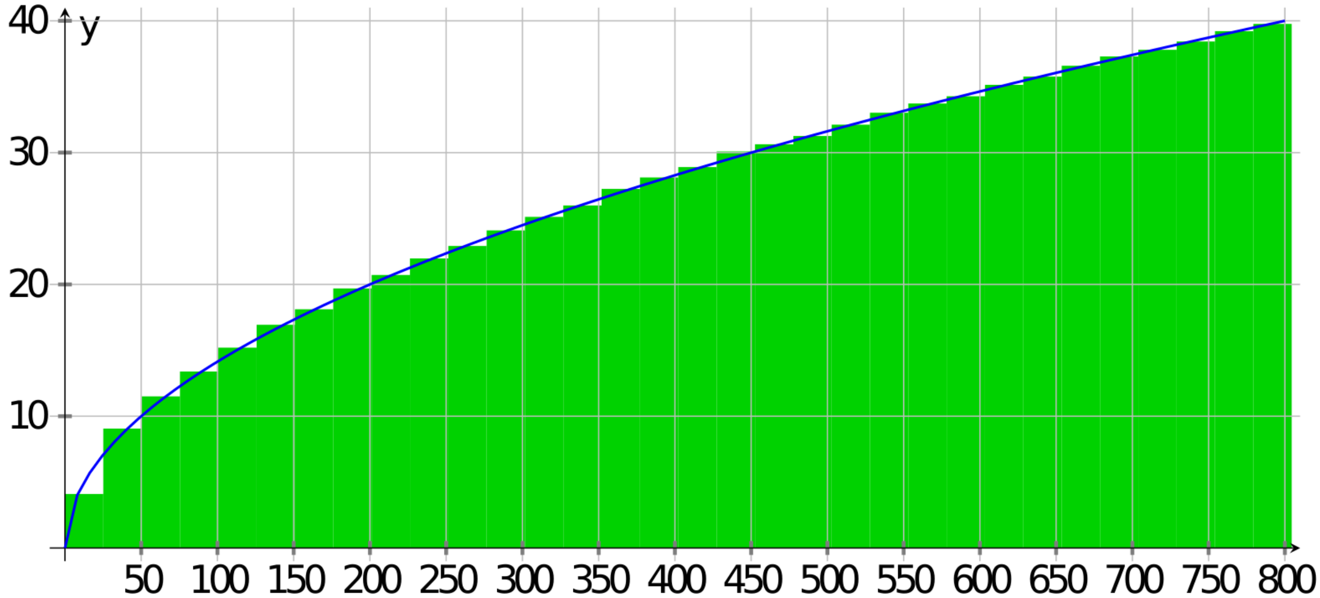
Time (s)	Velocity given by $f(t) = \sqrt{2t}$ in m/s
0	0
100	14.1
200	20
300	24.5
400	28.3
500	31.6
600	34.6
700	37.4
800	40

You can see the train spends much more time above 20 m/s than it does below, so that the train probably goes farther than $800 \cdot 20 = 16000$ meters (16 km).

What about the speed at time $t = 400$ which is 28.3 m/s? Can we use this value? Well, let's see what that would mean visually. That's basically treating the speed at the half-way point like it is a constant velocity:



So if we use a velocity of 28.3 meters per second, that would give a final distance of $800 \cdot 28.3 \approx 22600$ meters (22.6 km). But even this isn't quite right. The train spends an equal amount of time slower than 28.3 and faster than 28.3, but the slow region (from time 0 to time 400) is slower than the fast region (time 400 to time 800) is fast. They don't exactly cancel, so likely the train won't quite reach 22.6 km. Somehow, instead, we need to take into account the speed of the train at every instant. Sound familiar? This is where an integral is going to come it. Basically if we take the speed of the train at a small time frame, and multiply by the time it is going that speed, you get a picture that looks like this:



This is of course just the area under the curve! So we use the fundamental theorem of calculus to find the actual answer:

$$\begin{aligned}
\int_0^{800} \sqrt{2t}dt &= \int_0^{800} \sqrt{2} \cdot \sqrt{t}dt \\
&= \int_0^{800} \sqrt{2} \cdot t^{1/2}dt \\
&= \sqrt{2} \cdot \frac{1}{3/2} \cdot t^{3/2}\Big|_0^{800} \\
&= \sqrt{2}\frac{1}{3/2}(800)^{3/2} - \sqrt{2}\frac{1}{3/2}(0)^{3/2} \\
&\approx 21300 \text{ m}
\end{aligned}$$

A quick check to see if this is reasonable: we're saying the train went 21.3 km in those 13 minutes. Seems a bit fast for a train but not impossible. It's less than our max of 32 km, but more than the guess of 16 km that we thought it would beat. Seems reasonable!

CHAPTER 67

HOMEWORK: INTEGRAL APPLICATIONS

1. A car's velocity follows the equation $v(t) = 10t - t^2$ feet per second from $t = 0$ to $t = 10$. How far does the car travel during this time period?
   ≈ 166.7 feet
   ans
2. A car's velocity follows the equation $v(t) = 10 - \sqrt{t}$ from $t = 0$ to $t = 100$. How far does the car travel from $t = 0$ to $t = 100$?
   ≈ 333.33 units
   ans
3. A car's acceleration follows the equation $a(t) = t$ from $t = 0$ to $t = 10$. Recall that acceleration is the derivative of velocity.

   a. Find a function $v(t)$ for the velocity at time t.
      $v(t) = \frac{1}{2}t^2$ (you could also add any constant to this and still have a valid answer.)
      ans
   b. How far does the car travel from $t = 0$ to $t = 10$?
      Need to compute $\int_{0^10} \frac{1}{2}t^2 dt \approx 166.7$ units.
      ans

4. An employee's wages start at $10,000 a year and quickly increase after that at a rate of 0.04 per year, continuously implemented. Thus, at year t, the employee makes

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10000e^{0.0277t} <!--%% Where did 0.0277 come from?-->

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   dollars per year.

   a. How much does the employee make per year at year 5?
      $\$11485.5$
      ans
   b. How much total does the employee make in the first five years?

$\int_0^5 10000e^{0.0277t}dt \approx \$53628$
ans

5. Water drains from a tub at a rate of $\sqrt{50-2t}$ gallons per minute, with t measured in minutes.

   a. How long does it take for the rate to drop to zero?
   $t = 25$
   ans
   b. How much total water has been lost at this point?
   $\int_0^{25} \sqrt{50-2t}dt \approx 69.28$
   ans

6. A biologist models elk growth rate as $G(t) = 5e^{0.02t}$ measured in elk per year.

   a. How fast is the elk growth rate changing at $t = 10$?
   0.122 elk per year per year
   ans
   b. How many elk were born in the first 20 years of this model?
   $\int_0^{20} 5e^{0.02t} \approx 123$
   ans
   c. Do a sensitivity analysis. Given a small change in 5, how does that affect the answer to part (a)? Given a small change in 0.02, how does that affect part (a)?

7. Let $G(t)$ be the rate at which GDP is growing measured in dollars per day. Match the symbols $G'(t)$, $G(t)$ and $\int_0^t G(t)dt$ to the following statements.

   a. This measures the rate that GDP growth is speeding up or slowing down.
   $G'(t)$
   ans
   b. This measures how much GDP has increased since the beginning of the year.
   $\int_0^t G(t)dt$
   ans
   c. This measure how quickly GDP is increasing.
   $G(t)$
   ans

8. The Greenland ice sheet is losing ice. It is estimated that it is losing ice at a rate of $f(t) = -0.5t^2 - 150$ gigatonnes per year, with t measured in years, and $t = 0$ representing 2010. How many gigatonnes of ice will the ice sheet lose from 2015 to 2025?
   $\int_5^{15} -0.5t^2 - 150dt \approx -2040$ gigatonnes.
   ans
9. Let $C(t)$ be the crime rate in the city of Gotham, with $C(t)$ measured in crimes per day, and t measured in days. Match $C(t)$, $C'(t)$, and $\int C(t)dt$ to the following.

   a. This function would tell you how many crimes are committed over the last 90 days.
   $\int C(t)dt$

ans

b. This function would tell you how many crimes per day were being committed 90 days ago.
$C(t)$
ans

c. This function will tell you how quickly the crime rate was increasing or decreasing 90 days ago.
$C'(t)$
ans

10. When blasting off from the earth into space, a rocket uses fuel at a rate of $f(t) = 5 + 100e^{-0.01t}$, where t is measured in seconds and $f(t)$ is measured in gallons per second.

a. How many gallons are used in a four-minute flight starting at $t = 0$.
$\int_0^4 5 + 100e^{-.01t} \approx 412$
ans

b. How many gallons are used in a two-minute flight starting at $t = 0$?
$\int_0^2 5 + 100e^{-0.01t} \approx 208$
ans

c. Should your answer for (b) be exactly half of the answer for part (a)? Why or why not?
No, since rockets don't use fuel at a constant rate.
ans

11. The amount of sun power that is available to a flower is given by $S(t) = 2.5\sin\left(\frac{\pi}{12}t\right) + 2.5$ kilojoules per hour. The flower can absorb energy at

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efficiency, meaning it can use or store about

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```

of the available sunlight energy. How much energy (in kilojoules) does the flower absorb in a 48-hour period?
6 kilojoules
ans

12. **Submarine Navigation**
Nuclear submarines spend months underwater with no access to GPS or similar navigation techniques. Instead, they use a "dead reckoning" approach where accelerometers are used to keep track of how fast they are moving, from which their position can be determined. A submarine starts not moving at all. Given the following list of accelerations, estimate how far the submarine has gone.

Day	Average Acceleration in miles / day^2
0	220
1	135
2	-150
3	0
4	0
5	200
6	0
7	-405
8	0

CHAPTER 68

INTEGRATION BY PARTS

Recall the product rule:

$$\frac{d}{dx}f \cdot g = fg' + gf'$$

If we use u and v instead of f and g, this becomes

$$\frac{d}{dx}u \cdot v = uv' + vu'.$$

Now let's integrate both sides and solve for uv'

$$\begin{aligned}
\frac{d}{dx}u \cdot v &= uv' + vu' \\
\int \frac{d}{dx}uv dx &= \int uv' + vu' dx \\
uv &= \int uv' dx + \int vu' dx \\
uv - \int vu' dx &= \int uv' dx,
\end{aligned}$$

Flipping this around, we have the inverse product rule, also called *integration by parts.*

$$\boxed{\int uv' dx = uv - \int u'v}$$

The tricky part is what to use as u, and what to use as v'. Here are some steps and guidelines to follow, but it takes some intuition building before you know how to use it sometimes, and some product integrals cannot be solved with integration by parts.

Integration by parts:

1. Think of your original integral as a product. Identify a function that is easy to integrate, and set it equal to v'. The other function should be something that will simplify nicely once you take the derivative.
2. Find u' (take the derivative of u) and find v (integrate v')
3. Using substitution, plug in the values for u, v, v and u' in the integration by parts formula.
4. This gives you another integral — hopefully this one is easier. If not, you may need to use u -substitution, or even integration by parts a second time.

Example **Integration by parts**

Problem Find $\int_0^2 xe^x dx$.

Let's follow the integration by parts steps:

1. The function we are integrating is xe^x, which is a product in two pieces: x and e^x. While x is easy to integrate, e^x is even nicer. We will start with $v' = e^x$, and $u = x$.
2. We see $u' = \frac{d}{dx}x = 1$, and $v = \int e^x dx = e^x$ (you don't need to worry about the $+C$ for now).
3. Using the formula with $u = x$, $u' = 1$, $v = e^x$, and $v' = e^x$, we have

$$\begin{aligned}\int_0^2 xe^x dx &= uv - \int u'v dx \\ &= (x)(e^x) - \int_0^2 (1)(e^x) dx \\ &= xe^x - \int_0^2 e^x dx\end{aligned}$$

4. We now have reduce the problem to an easier one: $\int e^x dx$. We continue:

$$\begin{aligned}\int_0^2 xe^x dx &= xe^x - \int_0^2 e^x dx \\ &= xe^x - e^x\Big|_0^2 \\ &= (2e^2 - e^2) - (0e^0 - e^0) \\ &= (e^2) - (0 - 1) \\ &= e^2 + 1\end{aligned}$$

So the answer is $e^2 + 1 \approx \boxed{8.389}$.

Notice if the problem contains an x variable, then this is usually a good choice for u since it will go away once you take the derivative u'.

Example **Integration by parts**

Problem Compute $\int(2x+3)\cos(x)dx$.

We following the steps of integration by parts.

1. Set $u=(2x+3)$, which simplifies nicely with the derivative, and let $v'=\cos(x)$ which is easy to integrate.
2. We have $u'=2$, and $v=\sin(x)$.
3. Applying the formula, we have

$$\begin{aligned}\int(2x+3)\cos(x)dx &= uv-\int u'vdx \\ &= (2x+3)\sin(x)-\int 2\sin(x)dx\end{aligned}$$

4. Continuing …

$$\begin{aligned}\int(2x+3)\cos(x)dx &= (2x+3)\sin(x)-\int 2\sin(x)dx \\ &= (2x+3)\sin(x)-2(-\cos(x)) \\ &= \boxed{(2x+3)\sin(x)+2\cos(x)+C}.\end{aligned}$$

Example **Integration by parts**

Problem Use integration by parts to find $\int x\sqrt{2x+1}dx$

This one will involve integration by parts and a u-substitution shortcut. Here are the steps of integration by parts:

1. We can integrate either function, but just like the previous case it's best to set $u=x$. This leaves $v'=\sqrt{2x+1}$.
2. We see $u'=\frac{d}{dx}x=1$. Notice $v=\int\sqrt{2x+1}dx$ is a bit harder — this is a u-sub shortcut though. Let's write it as $\int(2x+1)^{1/2}dx$. Using the inverse power rule, and remembering the $\frac{1}{m}$ factor from the u-sub shortcut, we have

$$v=\int(2x+1)^{1/2}dx=\tfrac{1}{2}\cdot\tfrac{(2x+1)^{3/2}}{3/2}=\tfrac{(2x+1)^{3/2}}{3}$$

3. Applying the formula with $u = x$, $u' = 1$, $v = \frac{(2x+1)^{3/2}}{3}$, and $v' = \sqrt{2x+1}$, we see

$$\begin{aligned}
\int x\sqrt{2x+1}dx &= uv - \int u'vdx \\
&= (x)\left(\frac{(2x+1)^{3/2}}{3}\right) - \int (1)\left(\frac{(2x+1)^{3/2}}{3}\right)dx \\
&= \frac{x(2x+1)^{3/2}}{3} - \int \frac{(2x+1)^{3/2}}{3}dx
\end{aligned}$$

At this point, doing the integral of $\int \frac{(2x+1)^{3/2}}{3}dx$ may not seem easy. However, don't fret — it's a power rule and u-sub shortcut problem. Watch.

$$\begin{aligned}
\frac{x(2x+1)^{3/2}}{3} - \int \frac{(2x+1)^{3/2}}{3}dx &= \frac{x(2x+1)^{3/2}}{3} - \frac{1}{3}\int (2x+1)^{3/2}dx \\
&= \frac{x(2x+1)^{3/2}}{3} - \frac{1}{3}\left(\frac{1}{2}\cdot\frac{(2x+1)^{5/2}}{5/2}\right) \\
&= \frac{x(2x+1)^{3/2}}{3} - \frac{1}{3}\left(\frac{1}{2}\cdot\frac{2(2x+1)^{5/2}}{5}\right) \\
&= \frac{x(2x+1)^{3/2}}{3} - \frac{1}{3}\left(\frac{(2x+1)^{5/2}}{5}\right) \\
&= \boxed{\frac{x(2x+1)^{3/2}}{3} - \frac{(2x+1)^{5/2}}{15} + C}
\end{aligned}$$

Chapter 69

HOMEWORK: INTEGRATION BY PARTS

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1. Solve each of the following using integration by parts:

    a. $\int x\cos(x)dx$
    $x\sin(x)+\cos(x)+C$
    ans
    b. $\int(4x-1)\cos(x)dx$
    $(4x-1)\sin(x)+4\cos(x)+C$
    ans
    c. $\int x\sin(x)dx.$
    $-x\cos(x)+\sin(x)+C$
    ans
    d. $\int xe^x dx.$
    xe^x-e^x+C
    ans
    e. $\int\ln(x)dx.$ (Hint: Let $u=\ln(x)$ and $v'=1$)
    $x\ln(x)-x$
    ans

2. Watch the following Khan Academy video: Integration by parts twice
3. Use integration by parts to solve $\int x^2\cos(x)dx.$
$x^2\sin(x)+2x\cos(x)-2\sin(x)+C$
ans
4. Use integration by parts to solve $\int x^3e^x dx.$
$x^3e^x-3x^2e^x+6xe^x-6e^x+C$
ans
5. Watch the following Khan Academy video: Integration by parts with e and cos together.
6. Use integration by parts to find $\int e^x\sin(x)dx.$
$\frac{\sin(x)e^x-\cos(x)e^x}{2}+C$
ans
7. Two part question:

    a. Use u-substitution to find $\int\sin(2x)dx$ and $\int\cos(2x)dx.$
    $-\frac{1}{2}\cos(2x)$ and $\frac{1}{2}\sin(2x)$
    ans
    b. Use integration by parts to find $\int x\sin(2x)dx.$

$-\frac{1}{2}x\cos(2x) - \frac{1}{4}\sin(2x)$
ans

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These resources were inspiration for parts of this text:

- *Calculus* by Gilbert Strang.
- *Calculus* by Michael Spivek
- KhanAcademy Calculus
- Calculus For the Life Sciences: A Modeling Approach by James L. Cornette and Raph A. Ackerman (Volume 1 and Volume 2)
- Montana State University Library Accessibility Checklist
- University of Maryland MathBench Biology Modules
- Better Explained: Calculus